AN EIGENFUNCTION STABILITY ESTIMATE FOR APPROXIMATE EXTREMALS OF THE L^p DYADIC MAXIMAL OPERATOR BELLMAN FUNCTION

ANTONIOS D. MELAS

Abstract. We prove a stability estimate for the functions that are almost extremals for the Bellman function related to the L^p norm of the dyadic maximal operator in the case $p \geq 2$. This estimate gives that such almost extremals are also almost "eigenfunctions" for the dyadic maximal operator, in the sense that the L^pdistance between the maximal operator applied to the function and a certain multiple of the function is small.

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1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is a useful tool in analysis and is defined by

$$M_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du : x \in Q,\ Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ where the dyadic cubes are the cubes formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, 2, ...$

As it is well known it satisfies the following L^p inequality

$$\|M_d\phi\|_p \leq \frac{p}{p-1}\|\phi\|_p \tag{1.2}$$

for every $p > 1$ and every $\phi \in L^p(\mathbb{R}^n)$ which has been proved best possible ([2] for the general martingales and [17] for dyadic ones).

In studying dyadic maximal operators as well as more general variants it would be convenient to work with functions supported in the unit cube $[0,1]^n$ and replace M_d by

$$M'_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du : x \in Q,\ Q \subseteq [0,1]^n \text{ is a dyadic cube}\right\} \tag{1.3}$$

and hence work completely on the measure space $[0,1]^n$ and actually we can work on a general nonatomic probability space with a martingale structure similar to the dyadic one.

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An approach for studying such maximal operators is the introduction of the so called Bellman functions (see [6]) related to them. It has been shown (see [3] and [10] for a different approach) that for any $p > 1$ the Bellman function

(1.4)
$$\mathcal{B}_p(F,f,L) = \sup\left\{\frac{1}{|Q|}\int_Q (M_d\phi)^p : \mathrm{Av}_Q(\phi^p) = F, \mathrm{Av}_Q(\phi) = f, \sup_{R:Q\subseteq R} \mathrm{Av}_R(\phi) = L\right\}$$

where Q is a fixed dyadic cube, R runs over all dyadic cubes containing Q, ϕ is nonnegative in $L^p(Q)$ and the variables F, f, L satisfy $0 \le f \le L, f^p \le F$, is given by

$$\mathcal{B}_p(F,f,L) = \begin{cases} F\omega_p\left(\dfrac{pL^{p-1}f-(p-1)L^p}{F}\right)^p \text{ if } L < \dfrac{p}{p-1}f \\ L^p + (\dfrac{p}{p-1})^p(F-f^p) \text{ if } L \ge \dfrac{p}{p-1}f. \end{cases} \tag{1.5}$$

where $\omega_p : [0,1] \to [1, \frac{p}{p-1}]$ denotes the inverse of the strictly decreasing function $H_p(z) = -(p-1)z^p + pz^{p-1}$ defined for $z \in [1, \frac{p}{p-1}]$. From this (1.2) follows, but the above gives more information on the behavior of the dyadic maximal operator, since it relates its size not only to the local L^p norm of the function ϕ but also to the local variance of it.

For more on Bellman functions and their relation to harmonic analysis we refer to [6], [7], [8] and [16]. For the exact evaluation of Bellman functions in certain cases we refer to [1], [2], [3], [5], [10], [11], [12], [13], [14], [15].

To prove (1.5) we let (X,μ) be a nonatomic probability space and let $\mathcal{T}$ be a family of measurable subsets of X that has a tree-like structure similar to the one in the dyadic case (see [3]) and we define the maximal operator associated to $\mathcal{T}$ as follows

$$M_{\mathcal{T}}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi|\, d\mu : x \in I \in \mathcal{T}\right\} \tag{1.6}$$

for every $\phi \in L^1(X,\mu)$. This can be viewed a local maximal operator, everything happening inside a unit cube i.e. X and is actually the corresponding martingale maximal operator. Then we consider the corresponding local type Bellman function

(1.7)
$$\mathcal{B}_p^{\mathcal{T}}(F,f) = \sup\{\int_X (M_{\mathcal{T}}\phi)^p d\mu : \phi \ge 0, \phi \in L^p(X,\mu), \int_X \phi^p d\mu = F, \int_X \phi d\mu = f\}.$$

for any $p > 1$ and in [3] we have proved the following, from which the other estimates follow:

$$\mathcal{B}_p^{\mathcal{T}}(F,f) = F\omega_p\left(\frac{f^p}{F}\right)^p. \tag{1.8}$$

In analyzing this estimate more deeply one is lead to consider what properties have the extremals or approximate extremals for it. In this direction it has been proved in [9] (for fixed F, f, p) that if a sequence (ϕ_n) of nonnegative functions is extremal for (1.8) in the sense that $\int_X \phi_n^p d\mu = F, \int_X \phi_n d\mu = f$ and $\lim_n \int_X (M_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p\left(\frac{f^p}{F}\right)^p$ then in the limit the sequence behaves like an approximate "eigenfunction" of $M_{\mathcal{T}}$ meaning that $\lim_n \int_X |M_{\mathcal{T}}\phi_n - c\phi_n|^p \, d\mu = 0$ where $c = \omega_p\left(\dfrac{f^p}{F}\right)$.

In the present paper we will provide for the case $p \geq 2$ a stability estimate that bounds the strong L^p-deviation of an almost extremal function from being an "eigenfunction"of $M_{\mathcal{T}}$. The precise statement is given by the following main theorem of this paper.

Theorem 1. *Let $p \geq 2$ be given. Then there exists an absolute constant C_p such that: If $(X, \mu, \mathcal{T})$ is a nonatomic probability space equipped with a tree-like family, if $F, f > 0$ are real numbers with $f < F^{1/p}$ and if $\delta > 0$ is sufficiently small then for any nonnegative function $\phi \in L^p(X)$ satisfying $\int_X \phi^p d\mu = F, \int_X \phi d\mu = f$ and $\int_X (M_{\mathcal{T}}\phi)^p d\mu \geq (1-\delta) F \omega_p\left(\dfrac{f^p}{F}\right)^p$ the following holds*

$$\int_X |M_{\mathcal{T}}\phi - c\phi|^p \, d\mu \leq C_p F \delta \tag{1.9}$$

where $c = \omega_p\left(\dfrac{f^p}{F}\right)$.

Using the properties of ω_p (see Lemma 2 in [3]) it is then easy to deduce the following, which provides in particular a stability estimate for the classical Doob's inequality (1.2) for martingales.

Corollary 1. *Let $p \geq 2$ be given. Then there exists a absolute constants A_p and B_p such that: If $(X, \mu, \mathcal{T})$ is a nonatomic probability space equipped with a tree-like family and if $\phi \in L^p(X)$ is a nonnegative function satisfying $\|M_d\phi\|_p \geq$*

$(\dfrac{p}{p-1} - \varepsilon)\|\phi\|_p$ where $\varepsilon > 0$ is sufficiently small then we have:

$$\|\phi\|_1 \leq A_p \|\phi\|_p \varepsilon \text{ and } \left\|M_{\mathcal{T}}\phi - \frac{p}{p-1}\phi\right\|_p \leq B_p \|\phi\|_p \varepsilon. \tag{1.10}$$

The proof of Theorem 1 uses the combinatorial approach for the Bellman function $\mathcal{B}_p^{\mathcal{T}}$, that was introduced in [3].

2. Preliminaries

As in [3] we let (X, μ) be a nonatomic probability space (i.e. $\mu(X) = 1$). We give the following.

Definition 1. *(a) A set $\mathcal{T}$ of measurable subsets of X will be called an N-homogeneous tree (where $N > 1$ is an integer) if the following conditions are satisfied:*

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ there corresponds a finite subset $\mathcal{C}(I) \subseteq \mathcal{T}$ containing N elements each having measure equal to $N^{-1}\mu(I)$ such that the elements of $\mathcal{C}(I)$ are pairwise disjoint subsets of I and $I = \bigcup \mathcal{C}(I)$.

(ii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} \mathcal{C}(I)$.

We could replace the disjointedness condition in (ii) above by asking that the pairwise intersections have measure 0 instead. But then one would replace X by $X \setminus \bigcup_{I \in \mathcal{T}} \bigcup_{J_1, J_2 \in \mathcal{C}(I),\ J_1 \neq J_2} (J_1 \cap J_2)$ which has full measure.

Now given any such $\mathcal{T}$ we define the maximal operator associated to it as follows

$$M_{\mathcal{T}}\psi(x) = \sup\{\mathrm{Av}_I(|\psi|) : x \in I \in \mathcal{T}\} \tag{2.1}$$

for every $\psi \in L^1(X,\mu)$ where for any nonnegative $\phi \in L^1(X,\mu)$ and for any $I \in \mathcal{T}$ we have written $\mathrm{Av}_I(\phi) = \frac{1}{\mu(I)}\int_I \phi d\mu$.

Let ϕ be a nonnegative nonconstant $\mathcal{T}$-step function, that is there exist an integer $m > 0$ and $\lambda_P \geq 0$ for each $P \in \mathcal{T}_{(m)}$ such that

$$\phi = \sum_{P \in \mathcal{T}_{(m)}} \lambda_P \chi_P \tag{2.2}$$

(where χ_P denotes the characteristic function of P). For every $x \in X$ we let $I_\phi(x)$ denote the unique *largest* element of the set $\{I \in \mathcal{T} : x \in I \text{ and } M_{\mathcal{T}}\,\phi(x) = \mathrm{Av}_I(\phi)\}$ (which is nonempty since $\mathrm{Av}_J(\phi) = \mathrm{Av}_P(\phi)$ whenever $P \in \mathcal{T}_{(m)}$ and $J \subseteq P$). Next for any $I \in \mathcal{T}$ we define the set

$$A_I = A(\phi, I) = \{x \in X : I_\phi(x) = I\} \tag{2.3}$$

and we let $\mathcal{S} = \mathcal{S}_\phi$ denote the set of all $I \in \mathcal{T}$ such that A_I is nonempty. It is clear that each such A_I is a union of certain P's from $\mathcal{T}_{(m)}$ and moreover

$$M_{\mathcal{T}}\,\phi = \sum_{I \in \mathcal{S}} \mathrm{Av}_I(\phi)\chi_{A_I}. \tag{2.4}$$

We define the correspondence $I \to I^*$ with respect to $\mathcal{S}$ as follows: for any $I \in \mathcal{S}$, I^* is the minimal element in the set of all $J \in \mathcal{S}$ that properly contain I. This is defined for every I in $\mathcal{S}$ that is not maximal with respect to $\subseteq$. We also write $y_I = \mathrm{Av}_I(\phi)$ for every $I \in \mathcal{S}$.

The main properties of the above are given in the following (see [3] for the proofs).

Lemma 1. *(i) For every $I \in \mathcal{S}$ we have $I = \bigcup_{\mathcal{S} \ni J \subseteq I} A_J$.*

(ii) For every $I \in \mathcal{S}$ we have $A_I = I \setminus \bigcup_{J \in \mathcal{S}: J^* = I} J$ and so $\mu(A_I) = \mu(I) - \sum_{J \in \mathcal{S}: J^* = I}\mu(J)$ and $\mathrm{Av}_I(\phi) = \frac{1}{\mu(I)}\sum_{J \in \mathcal{S}: J \subseteq I}\int_{A_J}\phi d\mu$.

(iii) For a $I \in \mathcal{T}$ we have $I \in \mathcal{S}$ if and only if $\mathrm{Av}_Q(\phi) < \mathrm{Av}_I(\phi)$ whenever $I \subseteq Q \in \mathcal{T}$, $I \neq Q$. In particular $X \in \mathcal{S}$ and so $I \to I^*$ is defined for all $I \in \mathcal{S}$ such that $I \neq X$.

The above Lemma shows that this linearization of $M_{\mathcal{T}}\,\phi$ may be viewed as a multiscale version of the classical Calderon-Zygmund decomposition.

We will also need the following technical Lemma (similar to the well known Clarkson's inequalities).

Lemma 2. *Let $p \geq 2$ be given. Then*

(i) For all $s, t \geq 0$ we have

$$t^p - s^p \geq |t - s|^p + p(t - s)s^{p-1}. \tag{2.5}$$

(ii) If (X,μ) is a nonatomic probability space and if $h \in L^p(X)$ is nonnegative then

$$\int_X h^p d\mu - (\int_X h d\mu)^p \geq \int_X \left|h - \int_X h d\mu\right|^p d\mu. \tag{2.6}$$

(iii) If $x, y, \lambda, \mu > 0$ then

$$\lambda x^p + \mu y^p - (\lambda+\mu)\left(\frac{\lambda x + \mu y}{\lambda+\mu}\right)^p \geq \frac{\lambda\mu(\lambda^{p-1}+\mu^{p-1})}{(\lambda+\mu)^p}\,|x-y|^p\,. \tag{2.7}$$

Proof. (i) By homogeneity it suffices to prove it when $s = 1$. If $t \geq 1$ we let $t = 1+x,\ x \geq 0$ and note that the function $f(x) = (1+x)^p - x^p - 1$ has $f''(x) = p(p-1)[(1+x)^{p-2} - x^{p-2}] \geq 0$ and $f(0) = 0, f'(0) = p$. If $t < 1$ we let $t = 1-x$ $(0 < x < 1)$ and note that $g(x) = (1-x)^p - x^p - 1 + px$ has $g'(x) = p(1-(1-x)^{p-1} - x^{p-1}) \geq 0$ since $0 < x < 1$ and $p \geq 2$.

(ii) This follows using (i) to get $h^p(x) - (\int_X h d\mu)^p \geq \left|h(x) - \int_X h d\mu\right|^p + p(h(x) - \int_X h d\mu)(\int_X h d\mu)^{p-1}$ and integrating $d\mu(x)$.

(iii) Follows from (ii) with h defined on $[0,1]$ (with the Lebesgue measure) to be x on $[0, \lambda/(\lambda+\mu)]$ and y on $[\lambda/(\lambda+\mu), 1]$. □

3. Proof of Theorem 1

Here we will denote by C any absolute positive constant, depending only on $p \geq 2$. First we remark that it suffices to prove Theorem 1 in the case where ϕ is supposed to be a nonnegative $\mathcal{T}$-step function. Indeed for a general ϕ let ϕ_m be its conditional expectation on $\mathcal{T}_{(m)}$ that is $\phi_m = \sum_{I\in\mathcal{T}_{(m)}} \mathrm{Av}_J(\phi)\chi_I$ and so $\int_X \phi_m d\mu = f$, $F_m = \int_x \phi_m^p d\mu \leq \int_x \phi^p d\mu = F$, by Holder, and moreover the sequence $M_\mathcal{T}\,\phi_m = \sum_{I\in\mathcal{T}_{(m)}} \max\{\mathrm{Av}_J(\phi) : I \subseteq J \in \mathcal{T}\}\chi_I$ converges monotonically to $M_\mathcal{T}\,\phi$ and so using (1.8) and the fact that $\frac{\omega_p(x)^p}{x}$ is striclty decreasing (see Lemma 2 in [3]) we get

$$(1-\delta)F\omega_p\left(\frac{f^p}{F}\right)^p \leq \int_X (M_\mathcal{T}\phi)^p d\mu \leq \lim_m \int_X (M_\mathcal{T}\phi_m)^p d\mu \leq$$
$$\leq \liminf_m F_m\omega_p\left(\frac{f^p}{F_m}\right)^p \leq F\omega_p\left(\frac{f^p}{F}\right)^p$$

thus $F - \liminf F_m \leq CF\delta$ and therefore using also Lemma 2(ii) on each $I \in \mathcal{T}_{(m)}$ (with normalized measure) and adding we have $\liminf \int_X |\phi - \phi_m|^p\,d\mu \leq F - \liminf F_m \leq CF\delta$ and $\int_X (M_\mathcal{T}\phi_m)^p d\mu \geq (1 - \dfrac{\delta}{2})F_m\omega_p\left(\dfrac{f^p}{F_m}\right)^p$ for all sufficiently large m and noting that

$$\left(\int_X |M_\mathcal{T}\phi - c\phi|^p\,d\mu\right)^{1/p} \leq \left(\int_X |M_\mathcal{T}\phi - M_\mathcal{T}\phi_m|^p\,d\mu\right)^{1/p} +$$
$$+c\left(\int_X |\phi - \phi_m|^p\,d\mu\right)^{1/p} + \left(\int_X |M_\mathcal{T}\phi_m - c\phi_m|^p\,d\mu\right)^{1/p}$$

we conclude that having the estimate of Theorem 1 for ϕ_m for all sufficiently large m we get also it for the general ϕ (with a different constant C_p).

Therefore we assume from now on that ϕ is a nonnegative $\mathcal{T}$-step function and use the decomposition (2.4) of $M_{\mathcal{T}}\,\phi$. Next we define the function

$$\psi = \sum_{I\in\mathcal{S}}(\frac{1}{\mu(A_I)}\int_{A_I}\phi d\mu)\chi_{A_I} \tag{3.1}$$

noting that by (2.4) and Lemma 1 $M_{\mathcal{T}}\phi \le M_{\mathcal{T}}\psi$ pointwise and so writing $\bar{F} = \int_X \psi^p d\mu \le \int_X \phi^p d\mu = F$ we have

$$(1-\delta)F\omega_p\left(\frac{f^p}{F}\right)^p \le \int_X (M_{\mathcal{T}}\phi)^p d\mu \le \int_X (M_{\mathcal{T}}\psi)^p d\mu \le \bar{F}\omega_p\left(\frac{f^p}{\bar{F}}\right)^p$$

hence $F-\bar{F}\le CF\delta$. Thus Lemma 2 (ii) gives

$$\int_X |\phi-\psi|^p\,d\mu = \sum_{I\in\mathcal{S}}\mu(A_I)\int_{A_I}\left|\phi-\frac{1}{\mu(A_I)}\int_{A_I}\phi d\mu\right|^p\frac{d\mu}{\mu(A_I)} \le$$

$$\le \sum_{I\in\mathcal{S}}\mu(A_I)\left[\int_{A_I}\phi^p\frac{d\mu}{\mu(A_I)} - \left(\frac{1}{\mu(A_I)}\int_{A_I}\phi d\mu\right)^p\right] = F-\bar{F}\le CF\delta \tag{3.2}$$

and since $M_{\mathcal{T}}\psi \le M_{\mathcal{T}}\phi + M_{\mathcal{T}}\,|\phi-\psi|$ it suffices to prove the estimate (1.9) with ψ instead of ϕ.

We now let as in [3], $a_I = \mu(A_I) = \mu(I) - \sum_{J\in\mathcal{S}:J^*=I}\mu(J)$, $\rho_I = \dfrac{a_I}{\mu(I)} \in (0,1)$ and $y_I = \mathrm{Av}_I(\phi)$ for every $I\in\mathcal{S}$ and then Lemma 1 implies that with $x_I = \frac{1}{\mu(A_I)}\int_{A_I}\phi d\mu$ we have

$$x_I = \frac{y_I\mu(I) - \sum_{J\in\mathcal{S}:J^*=I} y_J\mu(J)}{\mu(I)-\sum_{J\in\mathcal{S}:J^*=I}\mu(J)}. \tag{3.3}$$

Moreover we let $c = \omega_p\left(\dfrac{f^p}{F}\right) > 1$, $\beta = c-1>0$, $\tau_I = \beta+1-\beta\rho_I > 0$ and using Hōlder's inequality as in [3] we get

$$F \ge \sum_{I\in\mathcal{S}}\frac{(y_I\mu(I)-\sum_{J^*=I}y_J\mu(J))^p}{(\mu(I)-\sum_{J^*=I}\mu(J))^{p-1}} \ge$$

$$\ge \sum_{I\in\mathcal{S}}(\frac{(y_I\mu(I))^p}{(\tau_I\mu(I))^{p-1}} - \sum_{J^*=I}\frac{(y_J\mu(J))^p}{((\beta+1)\mu(J))^{p-1}}) =$$

$$= \sum_{I\in\mathcal{S}}\frac{(y_I\mu(I))^p}{(\tau_I\mu(I))^{p-1}} - \sum_{\substack{I\in\mathcal{S}\\ I\neq X}}\frac{(y_I\mu(I))^p}{((\beta+1)\mu(I))^{p-1}} =$$

$$= \frac{y_X^p}{\tau_X^{p-1}} + \sum_{\substack{I\in\mathcal{S}\\ I\neq X}}\frac{1}{\rho_I}(\frac{1}{\tau_I^{p-1}} - \frac{1}{(\beta+1)^{p-1}})a_I y_I^p =$$

$$= \frac{y_X^p}{\tau_X^{p-1}} + \sum_{\substack{I\in\mathcal{S}\\ I\neq X}}\frac{1}{\rho_I}(\frac{1}{(\beta+1-\beta\rho_I)^{p-1}} - \frac{1}{(\beta+1)^{p-1}})a_I y_I^p \ge$$

$$\ge \frac{f^p}{\tau_X^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p}\sum_{\substack{I\in\mathcal{S}\\ I\neq X}} a_I y_I^p + \frac{(p-1)p\beta^2}{2(\beta+1)^{p+1}}\sum_{\substack{I\in\mathcal{S}\\ I\neq X}}\rho_I a_I y_I^p \tag{3.4}$$

since for $x>0$

$$\frac{1}{(\beta+1-\beta x)^{p-1}}-\frac{1}{(\beta+1)^{p-1}}=\int_0^x\frac{(p-1)\beta}{(\beta+1-\beta u)^p}du\geq$$

$$\int_0^x\left[\frac{(p-1)\beta}{(\beta+1)^{p+1}}+\frac{(p-1)p\beta^2u}{(\beta+1-\beta u)^{p+1}}\right]du\geq\frac{(p-1)\beta x}{(\beta+1)^p}+\frac{(p-1)p\beta^2x^2}{2(\beta+1)^{p+1}} \tag{3.5}$$

Therefore we have (using also (3.5) with $x=\rho_X$)

$$F\geq\frac{1}{(\beta+1)^{p-1}}f^p+\frac{(p-1)\beta}{(\beta+1)^p}\int_X(M_{\mathcal{T}}\phi)^pd\mu+\frac{(p-1)p\beta^2}{2(\beta+1)^{p+1}}\int_X\rho(M_{\mathcal{T}}\phi)^pd\mu \tag{3.6}$$

where $\rho=\sum_{I\in\mathcal{S}}\rho_I\chi_I$. But since as can be easily seen $(1+\frac{1}{\beta})\frac{(\beta+1)^{p-1}F-f^p}{p-1}=F\omega_p\left(\frac{f^p}{F}\right)^p$ the assumption $\int_X(M_{\mathcal{T}}\phi)^pd\mu\geq(1-\delta)F\omega_p\left(\frac{f^p}{F}\right)^p$ combined with (3.6) gives

$$F\geq F-CF\delta+C\beta^2\int_X\rho(M_{\mathcal{T}}\phi)^pd\mu. \tag{3.7}$$

thus

$$(c-1)^2\int_X\rho(M_{\mathcal{T}}\phi)^pd\mu\leq CF\delta. \tag{3.8}$$

Now ignoring the term leading to $\int_X\rho(M_{\mathcal{T}}\phi)^pd\mu$ in the string of inequalities (3.4) resulting in (3.7) we get (using Holder in the first step)

$$CF\delta\geq\sum_{I\in\mathcal{S}}\left[\frac{(y_I\mu(I)-\sum_{J^*=I}y_J\mu(J))^p}{(\mu(I)-\sum_{J^*=I}\mu(J))^{p-1}}-\frac{(y_I\mu(I))^p}{(\tau_I\mu(I))^{p-1}}+\sum_{J^*=I}\frac{(y_J\mu(J))^p}{(c\mu(J))^{p-1}}\right]\geq$$
$$\geq\sum_{I\in\mathcal{S}}\left[a_Ix_I^p-\frac{(y_I\mu(I))^p}{(\tau_I\mu(I))^{p-1}}+\frac{(\sum_{J^*=I}y_J\mu(J))^p}{(\sum_{J^*=I}c\mu(J))^{p-1}}\right]=$$
$$=\sum_{I\in\mathcal{S}}\left[a_Ix_I^p+\frac{(y_I\mu(I)-a_Ix_I)^p}{(c\mu(I)-ca_I)^{p-1}}-\frac{(y_I\mu(I))^p}{(c\mu(I)-(c-1)a_I)^{p-1}}\right]$$

and since by Lemma 2 (iii)

$$\rho_Ix_I^p+c(1-\rho_I)\left(\frac{y_I-\rho_Ix_I}{c(1-\rho_I)}\right)^p-(c-(c-1)\rho_I)\left(\frac{y_I}{c-(c-1)\rho_I}\right)^p\geq$$
$$\geq\rho_Ic(1-\rho_I)\frac{\rho_I^{p-1}+(c(1-\rho_I))^{p-1}}{(c-(c-1)\rho_I)^p}\left|x_I-\frac{y_I-\rho_Ix_I}{c(1-\rho_I)}\right|^p\geq$$
$$\geq C\rho_I\left|(c-(c-1)\rho_I)x_I-y_I\right|^p$$

we conclude that

$$\int_X\left|M_{\mathcal{T}}\phi-c\psi+(c-1)\rho\psi\right|^pd\mu=\sum_{I\in\mathcal{S}}a_I\left|(c-(c-1)\rho_I)x_I-y_I\right|^p\leq CF\delta.$$

But by (3.8)

$$\int_X\left|(c-1)\rho\psi\right|^pd\mu\leq(c-1)^p\int_X\rho\psi^pd\mu\leq C(c-1)^2\int_X\rho(M_{\mathcal{T}}\psi)^pd\mu\leq CF\delta$$

so using also $M_{\mathcal{T}}\psi\leq M_{\mathcal{T}}\phi+M_{\mathcal{T}}|\phi-\psi|$ and (3.2) we get the desired estimate for ψ and this completes the proof.

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Department of Mathematics, University of Athens, Panepistimiopolis 15784, Athens, Greece

E-mail address: amelas@math.uoa.gr