PRECOVERS AND ORTHOGONALITY IN THE STABLE MODULE CATEGORY

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Abstract. We show that any module admits a presentation as the quotient of a Gorenstein projective module by a submodule which is itself right orthogonal, with respect to the standard Ext^1 pairing, to the class of Gorenstein projective modules of type FP_∞. For that purpose, we use the concept of orthogonality in the stable module category and examine the orthogonal pair which is induced therein by the class of completely finitary Gorenstein projective modules.

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0. Introduction

Auslander has introduced the finitely generated modules of Gorenstein dimension zero over a commutative Noetherian ring in [2] and [3], as a generalization of the finitely generated projective modules, in order to study the finer homological properties of these rings. Since then, this concept has found many applications in commutative algebra and algebraic geometry; it turns out that the properties of modules of Gorenstein dimension zero are very closely related to the structure of the singularities of a Gorenstein ring. The generalization of this notion to any (not necessarily finitely generated) module over any (not necessarily commutative Noetherian) ring R by Enochs and Jenda in [13] lead to the definition of Gorenstein projective modules: These are precisely the syzygy modules of the complete projective resolutions (a.k.a. totally acyclic complexes of projective modules), i.e. of the doubly infinite acyclic complexes of projective modules

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots , \tag{1}$$

which remain acyclic after applying the functor $\text{Hom}_R(\_, P)$ for any projective module P. The class $\texttt{GP}(R)$ of Gorenstein projective modules has several interesting properties, as shown by Holm in [17]. In particular, modules of finite Gorenstein projective dimension can be defined in the standard way, by using resolutions by Gorenstein projective modules.

Avramov and Martsinkovsky studied in [1] the relative cohomology of finitely generated modules of finite Gorenstein projective dimension over a two-sided Noetherian ring. This has been taken up by Holm [17], who showed that the Gorenstein Ext functors (i.e. the relative

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derived functors of Hom, with respect to the class of Gorenstein projective modules) can be always defined on the subcategory of modules of finite Gorenstein projective dimension. The key property that modules M in that subcategory enjoy is the existence of *proper* Gorenstein projective resolutions: Such a module M admits an exact sequence of the form

$$\cdots \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where:

(i) the modules G_i are Gorenstein projective for all $i \geq 0$ and

(ii) the sequence remains exact after applying the functor $\operatorname{Hom}_R(G, \_)$ for any Gorenstein projective module G.

It is through condition (ii) above that one can guarantee the independence of the Gorenstein Ext groups upon the choice of the particular resolution. The existence of proper Gorenstein projective resolutions in turn follows since any module M of finite Gorenstein projective dimension fits into an exact sequence

$$0 \longrightarrow K \longrightarrow G \xrightarrow{p} M \longrightarrow 0, \tag{2}$$

where G is a Gorenstein projective module and K has finite projective dimension. Then, the functor $\operatorname{Ext}^1_R(\_, K)$ vanishes on Gorenstein projective modules and hence the additive map

$$p_* : \operatorname{Hom}_R(G', G) \longrightarrow \operatorname{Hom}_R(G', M), \tag{3}$$

which is induced by the linear map p, is surjective for any Gorenstein projective module G'.

In general, we say that a module M admits a Gorenstein projective precover (a.k.a. right $\mathtt{GP}(R)$ approximation) if there exists a surjective linear map p from a Gorenstein projective module G onto M, for which the additive map (3) is surjective for all Gorenstein projective modules G'. Any surjective map p as above, with kernel K such that the functor $\operatorname{Ext}^1_R(\_, K)$ vanishes on Gorenstein projective modules, is a Gorenstein projective precover; such a precover is called special. If all modules admit a Gorenstein projective precover, then we say that the class $\mathtt{GP}(R)$ is precovering. Therefore, in order to define the Gorenstein Ext groups in the category of all modules, one has to show that $\mathtt{GP}(R)$ is precovering.

Besides the case of rings over which all modules have finite Gorenstein projective dimension, the class $\mathtt{GP}(R)$ is also known to be precovering over commutative Noetherian rings of finite Krull dimension. In fact, Jorgensen proved in [18] that a sufficient condition for the existence of Gorenstein projective precovers for any module is that the homotopy category of complete projective resolutions be a reflective subcategory of the full homotopy category of projective modules. It is shown in [loc.cit.] that this condition holds over any commutative Noetherian ring that admits a dualizing complex; subsequently, Murfet and Salarian showed in [22] that for the same condition to hold it only suffices to assume that the Noetherian ring has finite Krull dimension.

We recall that a module of type FP_∞ is a module that admits a projective resolution which consists of modules that are finitely generated (and projective) in each degree. It is easily seen that the Gorenstein projective modules of type FP_∞ are precisely the syzygy modules of the complete projective resolutions (1), where the projective modules P_n are finitely generated for all n. As a contribution to the existence problem of (special) Gorenstein projective precovers, we shall prove in this paper that any module M over any ring R admits a presentation as in (2), where G is Gorenstein projective and K is a module such that $\operatorname{Ext}^1_R(G', K)$ is the trivial group for all Gorenstein projective modules G' of type FP_∞. In fact, we prove something more general than this, by considering the so-called completely finitary modules. Here, we say that a module C is completely finitary if the complete cohomology functors $\widehat{\mathrm{Ext}}^*_R(C, \_)$, which were defined by Mislin [21], Benson and Carlson [7] and Vogel [16], commute with filtered colimits. The class of completely finitary modules includes the modules of type FP_∞ and the modules of finite projective dimension, whereas it is closed under extensions, kernels of epimorphisms, cokernels of monomorphisms and direct summands. In the special case of the integral group ring of certain hierarchically decomposable groups, completely finitary Gorenstein projective modules were studied in [10], in connection with the existence of Eilenberg-Mac Lane spaces that have finitely many cells in all sufficiently large dimensions.

We can now state the main result of this paper.

Theorem. *Let R be a ring and M an R-module. Then, there exists a short exact sequence*

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where G is a Gorenstein projective module and K is such that the functor $Ext^1_R(\_, K)$ vanishes on all completely finitary Gorenstein projective modules.

In general, the notion of precovering classes of modules (a.k.a. contravariantly finite subcategories) has been introduced by Auslander and Smalo [5] and, independently, by Enochs [12]. This notion has been extensively studied in the literature, mainly due to its importance in the representation theory of Artin algebras; see, for example, [4]. A useful tool for proving the existence of precovers is provided by the concept of cotorsion pairs, which was itself introduced by Salce in [24] (for abelian groups); the definition of cotorsion pairs is based on the orthogonality relation induced by the Ext^1 pairing of modules. Let us provisionally say that a class $\mathfrak{F}$ of modules admits a set of test modules for Ext^1 if there exists a *set* of modules $\mathfrak{X}$, such that the following two conditions are equivalent for any module M:

(i) $\mathrm{Ext}^1_R(F, M) = 0$ for all $F \in \mathfrak{F}$ and

(ii) $\mathrm{Ext}^1_R(X, M) = 0$ for all $X \in \mathfrak{X}$.

Using an elegant argument, Eklof and Trlifaj proved in [11] that any class $\mathfrak{F}$ which is part of a cotorsion pair $(\mathfrak{F}, \mathfrak{C})$ is precovering, provided that it admits a set of test modules for Ext^1. In this way, it is shown in [8] that the class of flat modules is precovering, using the flat cotorsion pair. One may also attempt to prove that certain classes of modules are precovering, by using an analogous approach, working with pairs of classes which are orthogonal with respect to the pairing provided by the Hom groups $\underline{\mathrm{Hom}}$ in the stable module category of the ring and not by the Ext^1 groups, as Eklof and Trlifaj did. In order to carry out such a programme, one has to argue that the class of modules under consideration admits a set of test modules for $\underline{\mathrm{Hom}}$ (with the obvious definition, in analogy with the case of Ext^1). At this point, the approach of considering $\underline{\mathrm{Hom}}$-orthogonal pairs, rather than Ext^1-orthogonal pairs, possesses an advantage: Conceptually, it is much easier to verify that a class admits a set of test modules for $\underline{\mathrm{Hom}}$; to solve the problem (at least locally, i.e. for one module at a time), we only have to exhibit a cofinal set of objects in a suitable comma category. This point is illustrated by Lemma 3.1 below; see also the discussion following its proof. On the other hand, for the classes of modules that we are interested in, the analogue of the result by Eklof and Trlifaj mentioned above is provided by an algebraic description of a certain homotopy colimit, which was motivated by algebraic topology and brought into the realm of algebra by Rickard [23] and Kropholler [19]. It is via this circle of ideas that we shall be able to prove our main result stated above.

Here is an outline of the contents of the paper: In Section 1, we collect certain preliminary notions and basic results and fix the notation that will be used throughout the paper. In the next section, we examine the hyperfinite extensions of Gorenstein projective modules and prove that these are Gorenstein projective as well. The class of stably finitely presented (Gorenstein projective) modules is shown in Section 3 to induce an orthogonal pair in the stable module category with some interesting properties. The main point here is to use the construction by Rickard and Kropholler, as an analogue of the technique used by Eklof and Trlifaj, in order to force the vanishing of certain Ext^1 groups in their work on cotorsion pairs with enough injective objects. Finally, in Section 4, we consider the class of completely finitary Gorenstein projective modules and prove our main result, by using a description of the modules in the (stable) right orthogonal of that class, in terms of the complete cohomology pairing.

Notations and terminology. All modules that are considered in this paper are left modules over a unital associative ring R. If X, Y and Z are three modules, then we shall identify the abelian group $\mathrm{Hom}_R(X, Y \oplus Z)$ with the direct sum $\mathrm{Hom}_R(X,Y) \oplus \mathrm{Hom}_R(X,Z)$; an element $(f,g) \in \mathrm{Hom}_R(X,Y) \oplus \mathrm{Hom}_R(X,Z)$ is then identified with the linear map $X \longrightarrow Y \oplus Z$, which is given by $x \mapsto (f(x), g(x))$, $x \in X$. There is an analogous identification of the abelian group $\mathrm{Hom}_R(X \oplus Y, Z)$ with the direct sum $\mathrm{Hom}_R(X,Z) \oplus \mathrm{Hom}_R(Y,Z)$; if $f \in \mathrm{Hom}_R(X,Z)$ and $g \in \mathrm{Hom}_R(Y,Z)$, then we denote by $[f,g] : X \oplus Y \longrightarrow Z$ the corresponding linear map, which is given by $(x,y) \mapsto f(x) + g(y)$, $(x,y) \in X \oplus Y$.

All direct systems in this paper are indexed by directed sets of indices. A functor F from the category of modules to that of abelian groups is said to commute with filtered colimits if for any direct system of modules $(M_i)_i$ the natural map $\varinjlim_i F(M_i) \longrightarrow F\left(\varinjlim_i M_i\right)$ is bijective.

1. Preliminaries

In this section, we recall a few notions and fix the notation that will be used throughout the paper.

If M, N are two modules, then the set consisting of those linear maps $M \longrightarrow N$ that factor through a projective module is a subgroup of the abelian group $\mathrm{Hom}_R(M,N)$. We denote by $\underline{\mathrm{Hom}}_R(M,N)$ the corresponding quotient group and let $[f]$ be the class of any linear map $f \in \mathrm{Hom}_R(M,N)$ therein. The composition of linear maps induces by passage to the quotients a well-defined biadditive $\underline{\mathrm{Hom}}$ pairing, that enables us to define the stable module category of the ring R as the additive category whose objects are all modules and whose morphism sets are given by the abelian groups $\underline{\mathrm{Hom}}_R(M,N)$.

It is well-known that a module M is finitely presented if and only if the functor $\mathrm{Hom}_R(M, \_)$ commutes with filtered colimits. We shall be interested in the stable analogue of this condition.

Definition 1.1. *A module M is called stably finitely presented if the functor $\underline{\mathrm{Hom}}_R(M, \_)$ commutes with filtered colimits. We denote by $\mathtt{SFP}(R)$ the class of all stably finitely presented modules.*

As it turns out, the stably finitely presented modules are precisely the retracts of the finitely presented modules in the stable module category.

Lemma 1.2. *The following conditions are equivalent for a module C:*

(i) C is stably finitely presented and

(ii) C is isomorphic to a direct summand of the direct sum $N \oplus P$ of two modules N and P, where N is finitely presented and P is projective. □

Proof. (i)→(ii): The argument is essentially that provided in [10, Lemma 1.5]. We express C as the colimit of a direct system $(C_i)_i$ of finitely presented modules and let $f_i : C_i \longrightarrow C$

be the canonical maps. Then, in view of our assumption on C, the natural map

$$\varinjlim_i \underline{\mathrm{Hom}}_R(C, C_i) \longrightarrow \underline{\mathrm{Hom}}_R(C, C)$$

is an isomorphism. Considering the identity map 1_C of C, it follows that there exists an index i and a linear map $g : C \longrightarrow C_i$, such that $[1_C] = [f_i g] \in \underline{\mathrm{Hom}}_R(C, C)$. The endomorphism $1_C - f_i g$ of C factors then through a suitable projective module P; in other words, there exist linear maps $a : C \longrightarrow P$ and $b : P \longrightarrow C$, such that $1_C - f_i g = ba$. The composition

$$C \xrightarrow{(g,a)} C_i \oplus P \xrightarrow{[f_i,b]} C$$

is therefore equal to $f_i g + ba = 1_C$ and hence C is a direct summand of $C_i \oplus P$.

(ii)→(i): Since the class $\mathtt{SFP}(R)$ is closed under direct summands, it only suffices to prove the result if C is equal to the direct sum of a finitely presented module and a projective module. Since the functor $\underline{\mathrm{Hom}}_R(P, \_)$ is identically zero if P is a projective module, we may reduce the problem to the case where the module C is finitely presented. In order to prove the result in this case, assume that C is finitely presented and let $(M_i)_i$ be a direct system of modules with structural maps $g_{ij} : M_i \longrightarrow M_j$, $i \leq j$. We also consider the colimit $M = \varinjlim_i M_i$ and the canonical maps $g_i : M_i \longrightarrow M$. Since C is finitely presented, the natural map

$$\nu : \varinjlim_i \mathrm{Hom}_R(C, M_i) \longrightarrow \mathrm{Hom}_R(C, M)$$

is bijective. It follows easily from the surjectivity of ν that the natural map

$$\underline{\nu} : \varinjlim_i \underline{\mathrm{Hom}}_R(C, M_i) \longrightarrow \underline{\mathrm{Hom}}_R(C, M)$$

is surjective. In order to prove that $\underline{\nu}$ is injective, we consider an index i and let $f : C \longrightarrow M_i$ be a linear map, such that $[g_i f] = [0] \in \underline{\mathrm{Hom}}_R(C, M)$. Then, the linear map $g_i f : C \longrightarrow M$ may be factored through a (projective and hence through a) free module F, as the composition $C \xrightarrow{a} F \xrightarrow{b} M$, for suitable linear maps a and b. Since C is finitely generated, we may assume that F is a finitely generated free module. Then, there exists an index $j \geq i$, such that the map b factors through M_j as the composition $F \xrightarrow{c} M_j \xrightarrow{g_j} M$ for a suitable linear map c. We now consider the linear map $g_{ij} f - ca : C \longrightarrow M_j$ and compute

$$g_j(g_{ij} f - ca) = g_j g_{ij} f - g_j ca = g_i f - ba = 0 \in \mathrm{Hom}_R(C, M).$$

Invoking the injectivity of ν, it follows that there is an index $k \geq j$, such that

$$g_{ik} f - g_{jk} ca = g_{jk}(g_{ij} f - ca) = 0 \in \mathrm{Hom}_R(C, M_k).$$

Then, we have $g_{ik} f = g_{jk} ca \in \mathrm{Hom}_R(C, M_k)$ and hence $[g_{ik} f] = [g_{jk} ca] = [0] \in \underline{\mathrm{Hom}}_R(C, M_k)$, where the latter equality follows from the projectivity of F. This shows that the image of $[f] \in \underline{\mathrm{Hom}}_R(C, M_i)$ vanishes in $\underline{\mathrm{Hom}}_R(C, M_k)$ and hence in the colimit $\varinjlim_i \underline{\mathrm{Hom}}_R(C, M_i)$. This completes the proof of the injectivity of $\underline{\nu}$. □

We now describe the notion of orthogonality in the stable module category that will be used in the paper. To that end, let $\mathfrak{X}$ be any class of modules. Then, the left orthogonal of $\mathfrak{X}$ is the class ${}^{\diamond}\mathfrak{X}$, consisting of those modules Y for which $\underline{\mathrm{Hom}}_R(Y, X) = 0$ for all $X \in \mathfrak{X}$. The right orthogonal of $\mathfrak{X}$ is the class $\mathfrak{X}^{\diamond}$, consisting of those modules Z for which $\underline{\mathrm{Hom}}_R(X, Z) = 0$ for all $X \in \mathfrak{X}$. This notion of orthogonality has several formal properties, which are analogous to those enjoyed by the more standard notion of orthogonality that is defined through the Ext^1 pairing. As an example, we note that $\mathfrak{X} \subseteq {}^{\diamond}(\mathfrak{X}^{\diamond})$ for any class $\mathfrak{X}$.

Another variation of the stable module category may be obtained by considering complete cohomology. Influenced by Gedrich and Gruenberg's work on terminal completions [14], Mislin has defined in [21] for any module M complete cohomology functors $\widehat{\mathrm{Ext}}^*_R(M,\_)$ and a natural transformation $\mathrm{Ext}^*_R(M,\_) \longrightarrow \widehat{\mathrm{Ext}}^*_R(M,\_)$, as the projective completion of the ordinary Ext functors. Equivalent definitions of complete cohomology have been independently formulated by Benson and Carlson in [7] and Vogel in [16].

These complete cohomology functors may be computed by means of certain resolutions, as we shall now describe. We say that a module M admits a complete projective resolution of coincidence index n if there exists a doubly infinite acyclic complex of projective modules, which remains acyclic after applying the functor $\mathrm{Hom}_R(\_,P)$ for any projective module P and coincides with a projective resolution of M in degrees $\geq n$. Then, the modules that admit a complete projective resolution of coincidence index 0 are precisely the Gorenstein projective modules. The class of these modules will be denoted by $\mathtt{GP}(R)$; it is closed under extensions, direct sums and direct summands (cf. [17]). Of course, all projective modules are Gorenstein projective.

It follows from Mislin's definition of complete cohomology that the groups $\widehat{\mathrm{Ext}}^*_R(M,N)$ may be computed, in the case where M admits a complete projective resolution P_* (of an arbitrary coincidence index), as the cohomology groups of the complex $\mathrm{Hom}_R(P_*,N)$ for any module N; cf. [9, Theorem 1.2]. In particular, if M is Gorenstein projective and N is any module, then the complete cohomology group $\widehat{\mathrm{Ext}}^0_R(M,N)$ may be naturally identified with $\underline{\mathrm{Hom}}_R(M,N)$ and the natural map $\mathrm{Ext}^i_R(M,N) \longrightarrow \widehat{\mathrm{Ext}}^i_R(M,N)$ is bijective for all $i \geq 1$.

Definition 1.3. *A module M is called completely finitary if the functors $\widehat{Ext}^i_R(M,\_)$ commute with filtered colimits for all i. We denote by $\mathtt{CF}(R)$ the class of all completely finitary modules.*

It follows readily that a completely finitary Gorenstein module is stably finitely presented. The class $\mathtt{CF}(R)$ contains all projective modules and is closed under direct summands, extensions, kernels of epimorphisms and cokernels of monomorphisms. It follows from [20, §4.1(ii)] that $\mathtt{CF}(R)$ contains all modules of type FP_∞, i.e. all modules that admit a projective resolution which consists of finitely generated (projective) modules in each degree.

2. Hyperfinite extensions of Gorenstein projective modules

As we have already noted above, Holm has shown in [17] that the class $\mathtt{GP}(R)$ of Gorenstein projective modules is closed under extensions. Using an inductive argument, it follows easily that an iterated extension of Gorenstein projective modules is Gorenstein projective as well. In other words, if n is a non-negative integer and $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n = M$ is an increasing filtration of a module M of length n, such that the quotient modules M_{i+1}/M_i are Gorenstein projective for all $i = 0, 1, \ldots, n-1$, then $M \in \mathtt{GP}(R)$. In this section, we shall obtain a version of that result for increasing filtrations of infinite length.

We begin with the following simple lemma. Its proof is certainly well-known, but we record it for the convenience of the reader.

Lemma 2.1. *(dual horseshoe lemma) Let*

$$0 \longrightarrow M' \stackrel{\iota}{\longrightarrow} M \stackrel{p}{\longrightarrow} M'' \longrightarrow 0$$

be a short exact sequence with M'' Gorenstein projective and assume that

$$0 \longrightarrow M' \stackrel{a}{\longrightarrow} P' \longrightarrow N' \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow M'' \stackrel{b}{\longrightarrow} P'' \longrightarrow N'' \longrightarrow 0$$

are two short exact sequences with P' projective. Then, there exists a short exact sequence

$$0 \longrightarrow M \xrightarrow{c} P' \oplus P'' \longrightarrow N \longrightarrow 0,$$

which fits into a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{\iota} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle a}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow & & \\
0 & \longrightarrow & P' & \longrightarrow & P' \oplus P'' & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(In the diagram above, the second row involves the natural embedding of P' into the direct sum and the natural projection of the direct sum onto P''.)

Proof. In view of our assumption on M'' and P', the abelian group $\mathrm{Ext}^1_R(M'', P')$ is trivial. It follows that the additive map

$$\iota^* : \mathrm{Hom}_R(M, P') \longrightarrow \mathrm{Hom}_R(M', P'),$$

which is induced by ι, is surjective. Therefore, there exists a linear map $a' : M \longrightarrow P'$, such that $a'\iota = a$. Then, the linear map $c = (a', bp) : M \longrightarrow P' \oplus P''$ fits into the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\iota} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle a}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow & & \\
0 & \longrightarrow & P' & \longrightarrow & P' \oplus P'' & \longrightarrow & P'' & \longrightarrow & 0
\end{array}$$

The argument is completed by letting $N = \operatorname{coker} c$ and invoking the snake lemma. □

Let $\mathfrak{C}$ be a class of modules. We say that a module M is a hyper-$\mathfrak{C}$ module (or a hyperfinite extension of modules of $\mathfrak{C}$) if there exists an ordinal number α and an ascending filtration of M by submodules M_β, which are indexed by the ordinals $\beta \leq \alpha$, such that $M_0 = 0$, $M_\alpha = M$ and $M_\beta / M_{\beta-1} \in \mathfrak{C}$ (resp. $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$) if $\beta \leq \alpha$ is a successor (resp. a limit) ordinal. In that case, we shall refer to the ascending chain of submodules $(M_\beta)_{\beta\leq\alpha}$ as a continuous ascending chain of submodules with sections in $\mathfrak{C}$. If $\alpha = 2$, this notion reduces to that of an extension of modules of $\mathfrak{C}$. We note that the direct sum of any family of modules which are contained in $\mathfrak{C}$ is a hyper-$\mathfrak{C}$ module. On the other hand, it is easily seen that any hyper-(hyper-$\mathfrak{C}$) module is a hyper-$\mathfrak{C}$ module. We state that property by saying that the class hyper-$\mathfrak{C}$ is closed under hyperfinite extensions; in particular, the class hyper-$\mathfrak{C}$ is closed under extensions.

A class $\mathfrak{C}$ of modules will be called Ω^{-1}-closed if for any $C \in \mathfrak{C}$ there exists a short exact sequence

$$0 \longrightarrow C \longrightarrow P \longrightarrow D \longrightarrow 0,$$

where P is projective and $D \in \mathfrak{C}$. As an example, we note that the class $\mathtt{GP}(R)$ is Ω^{-1}-closed.

Proposition 2.2. *Let R be a ring and consider an Ω^{-1}-closed class $\mathfrak{C}$ consisting of Gorenstein projective modules. We also consider a hyper-$\mathfrak{C}$ module M, which is endowed with a continuous ascending chain of submodules $(M_\beta)_{\beta\leq\alpha}$ with sections in $\mathfrak{C}$, for some ordinal number α. Then, there exists a hyper-$\mathfrak{C}$ module N with a continuous ascending chain of submodules $(N_\beta)_{\beta\leq\alpha}$*

with sections in $\mathfrak{C}$ and a family of projective modules $(P_\beta)_{\beta\leq\alpha}$, such that there is a short exact sequence

$$0 \longrightarrow M \longrightarrow Q \longrightarrow N \longrightarrow 0,$$

where $Q = \bigoplus_{\beta\leq\alpha} P_\beta$, and short exact sequences

$$0 \longrightarrow M_{\beta+1}/M_\beta \longrightarrow P_{\beta+1} \longrightarrow N_{\beta+1}/N_\beta \longrightarrow 0$$

for all $\beta < \alpha$.

Proof. We shall construct a continuous ascending chain of modules $(N_\beta)_{\beta\leq\alpha}$ with sections in $\mathfrak{C}$ and a family of projective modules $(P_\beta)_{\beta\leq\alpha}$, in such a way that $P_0 = N_0 = 0$ and there are short exact sequences of modules

$$0 \longrightarrow M_\beta \xrightarrow{\iota_\beta} \bigoplus\nolimits_{\gamma\leq\beta} P_\gamma \longrightarrow N_\beta \longrightarrow 0 \tag{4}$$

for all $\beta \leq \alpha$, which are compatible with each other, in the sense that for any two ordinal numbers β, β' with $\beta < \beta' \leq \alpha$ the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_\beta & \xrightarrow{\iota_\beta} & \bigoplus_{\gamma\leq\beta} P_\gamma & \longrightarrow & N_\beta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{\beta'} & \xrightarrow{\iota_{\beta'}} & \bigoplus_{\gamma\leq\beta'} P_\gamma & \longrightarrow & N_{\beta'} & \longrightarrow & 0 \end{array} \tag{5}$$

whose vertical arrows are the inclusion maps, is commutative. In particular, if β is an ordinal number with $\beta < \alpha$, then we may let $\beta' = \beta + 1$ in the above diagram and invoke the snake lemma in order to deduce the existence of a short exact sequence

$$0 \longrightarrow M_{\beta+1}/M_\beta \xrightarrow{j_{\beta+1}} P_{\beta+1} \longrightarrow N_{\beta+1}/N_\beta \longrightarrow 0,$$

where $j_{\beta+1}$ is the linear map which is induced from $i_{\beta+1}$ by passage to the quotients.

We shall proceed by transfinite induction on β and note that there is nothing to show for $\beta = 0$. We now let β be an ordinal number with $0 < \beta \leq \alpha$ and assume that the construction has been performed for all ordinals $\gamma < \beta$.

If β is a limit ordinal, then we define $P_\beta = 0$ and let $N_\beta = \varinjlim_{\gamma<\beta} N_\gamma$. Since we also have $M_\beta = \bigcup_{\gamma<\beta} M_\gamma = \varinjlim_{\gamma<\beta} M_\gamma$, the short exact sequences

$$0 \longrightarrow M_\gamma \xrightarrow{\iota_\gamma} \bigoplus\nolimits_{\delta\leq\gamma} P_\delta \longrightarrow N_\gamma \longrightarrow 0,$$

$\gamma < \beta$, induce (in view of the compatibility condition described by the commutative diagrams (5) and the fact that $\varinjlim_{\gamma<\beta} \bigoplus_{\delta\leq\gamma} P_\delta = \bigoplus_{\delta<\beta} P_\delta = \bigoplus_{\delta\leq\beta} P_\delta$) a short exact sequence

$$0 \longrightarrow M_\beta \xrightarrow{\iota_\beta} \bigoplus\nolimits_{\delta\leq\beta} P_\delta \longrightarrow N_\beta \longrightarrow 0.$$

The compatibility condition is obviously preserved by this definition.

We now assume that $\beta = \gamma + 1$ is a successor ordinal and consider the projective module $Q_\gamma = \bigoplus_{\delta\leq\gamma} P_\delta$ and the short exact sequence

$$0 \longrightarrow M_\gamma \xrightarrow{\iota_\gamma} Q_\gamma \longrightarrow N_\gamma \longrightarrow 0,$$

that has already been constructed. Since the quotient module $M_{\gamma+1}/M_\gamma$ is contained in the Ω^{-1}-closed class $\mathfrak{C}$, there exists a short exact sequence

$$0 \longrightarrow M_{\gamma+1}/M_\gamma \longrightarrow P_{\gamma+1} \longrightarrow C \longrightarrow 0,$$

where $P_{\gamma+1}$ is projective and $C \in \mathfrak{C}$. The $\mathfrak{C}$-module $M_{\gamma+1}/M_\gamma$ being Gorenstein projective, we may apply Lemma 2.1 and conclude that there exists a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_\gamma & \longrightarrow & M_{\gamma+1} & \longrightarrow & M_{\gamma+1}/M_\gamma & \longrightarrow & 0 \\ & & {\scriptstyle i_\gamma}\downarrow & & {\scriptstyle i_{\gamma+1}}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q_\gamma & \longrightarrow & Q_\gamma \oplus P_{\gamma+1} & \longrightarrow & P_{\gamma+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_\gamma & \longrightarrow & N_{\gamma+1} & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The cokernel of the monomorphism $N_\gamma \longrightarrow N_{\gamma+1}$ is then identified with the $\mathfrak{C}$-module C and this completes the inductive step of the construction. □

Remarks 2.3. (i) Keeping the same notation as in Proposition 2.2 and its proof, we note that the projective modules P_β where chosen therein to vanish when β is a limit ordinal. Hence, the projective module $Q = \bigoplus_{\beta\leq\alpha} P_\beta$ may be endowed with a continuous ascending chain of submodules $(Q_\beta)_{\beta\leq\alpha}$ with sections in the class of projective modules, by letting $Q_\beta = \bigoplus_{\gamma\leq\beta} P_\gamma$ for all $\beta \leq \alpha$. Moreover, the short exact sequence

$$0 \longrightarrow M \longrightarrow Q \longrightarrow N \longrightarrow 0$$

is compatible with the filtrations that are defined on the modules M, Q and N; this is another way of formulating the existence of the short exact sequences (4). Finally, the induced short exact sequence

$$0 \longrightarrow M_{\beta+1}/M_\beta \longrightarrow Q_{\beta+1}/Q_\beta \longrightarrow N_{\beta+1}/N_\beta \longrightarrow 0$$

reduces to the short exact sequence

$$0 \longrightarrow M_{\beta+1}/M_\beta \longrightarrow P_{\beta+1} \longrightarrow N_{\beta+1}/N_\beta \longrightarrow 0$$

in the statement of the proposition for all $\beta < \alpha$.

(ii) Proposition 2.2 implies that if $\mathfrak{C}$ is an Ω^{-1}-closed class consisting of Gorenstein projective modules, then the class hyper-$\mathfrak{C}$ is Ω^{-1}-closed as well.

(iii) A class $\mathfrak{C}$ of modules is called Ω^1-closed if for any $C \in \mathfrak{C}$ there exists a short exact sequence

$$0 \longrightarrow D \longrightarrow P \longrightarrow C \longrightarrow 0,$$

where the module P is projective and $D \in \mathfrak{C}$. For later use, we note that for any Ω^1-closed class $\mathfrak{C}$, the class hyper-$\mathfrak{C}$ is also Ω^1-closed.
Indeed, we may prove the analogue of Proposition 2.2 for Ω^1-closed classes using exactly the same arguments. (In fact, we only need the standard horseshoe lemma for the inductive step of the construction; hence, there is no need to assume that the sections of the ascending chains of submodules are Gorenstein projective.)

We shall use the following simple lemma, in order to prove that hyper-$\mathtt{GP}(R)$ modules are Gorenstein projective.

Lemma 2.4. *Let R be a ring and consider an Ω^{-1}-closed class $\mathfrak{C}$. If $Ext^i_R(C, P) = 0$, whenever $C \in \mathfrak{C}$, P is projective and $i > 0$, then $\mathfrak{C}$ consists of Gorenstein projective modules.*

Proof. Since the class $\mathfrak{C}$ is Ω^{-1}-closed, for any $\mathfrak{C}$-module C there exists an exact sequence

$$0 \longrightarrow C \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots \longrightarrow P_{-n} \longrightarrow \cdots ,$$

such that the module P_{-n} is projective and the image $\mathrm{im}\,(P_{-n} \longrightarrow P_{-n-1})$ is contained in $\mathfrak{C}$ for all $n \geq 1$. Splicing that exact sequence with any projective resolution of C, we may obtain a complete projective resolution of C of coincidence index 0. The existence of the latter shows that $C \in \mathtt{GP}(R)$, as needed. □

The following result has been proved in [25, Corollary 2.5], using arguments that are different than those employed here.

Corollary 2.5. *Any hyper-*$\mathtt{GP}(R)$ *module is Gorenstein projective.*

Proof. Since the functors $\mathrm{Ext}^i_R(\_\,, P)$ vanish on the class of Gorenstein projective modules for all projective modules P and all $i > 0$, we may invoke Auslander's lemma [11, Lemma 1] and conclude that these functors vanish on the class of hyper-$\mathtt{GP}(R)$ modules as well. On the other hand, Proposition 2.2, applied for the class $\mathtt{GP}(R)$ of all Gorenstein projective modules, implies that the class of hyper-$\mathtt{GP}(R)$ modules is Ω^{-1}-closed; cf. Remark 2.3(ii). The result is therefore a consequence of Lemma 2.4. □

Corollary 2.6. *Let R be a ring and consider a module M, which is endowed with a continuous ascending chain of submodules $(M_\beta)_{\beta\leq\alpha}$ with sections in $\mathtt{GP}(R)$, for some ordinal number α. If $i \in \mathbb{Z}$ and L is a module, such that $\widehat{Ext}^i_R(M_{\beta+1}/M_\beta, L) = 0$ for all $\beta < \alpha$, then we also have $\widehat{Ext}^i_R(M, L) = 0$.*

Proof. The modules $M_{\beta+1}/M_\beta$ are Gorenstein projective for all $\beta < \alpha$; in view of Corollary 2.5, this is also the case for the module M. Hence, if $i > 0$, the complete cohomology groups that appear in the statement coincide with the corresponding ordinary Ext groups. In the case where $i > 0$, the result then follows from Auslander's lemma. In order to prove the result in the case where $i \leq 0$, it suffices to show that for any non-negative integer j the group $\widehat{\mathrm{Ext}}^{1-j}_R(M, L)$ is trivial, whenever M is a module endowed with a continuous ascending chain of submodules $(M_\beta)_{\beta\leq\alpha}$ with sections in $\mathtt{GP}(R)$ and L is a module, such that $\widehat{\mathrm{Ext}}^{1-j}_R(M_{\beta+1}/M_\beta, L) = 0$ for all $\beta < \alpha$. We shall prove the latter claim by induction on j. As we noted above, the case where $j = 0$ follows from Auslander's lemma. We now assume that $j > 0$ and the result is known for $j - 1$. Given two modules M and L, as in the statement of the claim to be proved, we use Proposition 2.2 for the special case where $\mathfrak{C} = \mathtt{GP}(R)$ therein, in order to find a module N together with a continuous ascending chain of submodules $(N_\beta)_{\beta\leq\alpha}$ with sections in $\mathtt{GP}(R)$ and a family of projective modules $(P_\beta)_{\beta\leq\alpha}$, such that there is a short exact sequence

$$0 \longrightarrow M \longrightarrow Q \longrightarrow N \longrightarrow 0, \tag{6}$$

where $Q = \bigoplus_{\beta\leq\alpha} P_\beta$, and short exact sequences

$$0 \longrightarrow M_{\beta+1}/M_\beta \longrightarrow P_{\beta+1} \longrightarrow N_{\beta+1}/N_\beta \longrightarrow 0 \tag{7}$$

for all $\beta < \alpha$. The existence of the short exact sequences (7) implies that

$$\widehat{\mathrm{Ext}}^{1-(j-1)}_R(N_{\beta+1}/N_\beta, L) = \widehat{\mathrm{Ext}}^{1-j}_R(M_{\beta+1}/M_\beta, L) = 0$$

for all $\beta < \alpha$. Hence, applying the inductive hypothesis to the modules N and L, we conclude that $\widehat{\mathrm{Ext}}^{1-(j-1)}_R(N, L) = 0$. The result follows, since the existence of the short exact sequence (6) implies that $\widehat{\mathrm{Ext}}^{1-j}_R(M, L) = \widehat{\mathrm{Ext}}^{1-(j-1)}_R(N, L)$. □

We shall use Corollary 2.6 in the sequel for the special case where $i = 0$ therein, in the form stated below.

Corollary 2.7. *Let R be a ring and consider a class $\mathfrak{C}$, which consists of Gorenstein projective modules. Then, any hyper-$\mathfrak{C}$ module is contained in ${}^{\diamond}(\mathfrak{C}^{\diamond})$.*

Proof. Let M be a hyper-$\mathfrak{C}$ module and consider a continuous ascending chain of submodules $(M_\beta)_{\beta\leq\alpha}$ of it with sections in $\mathfrak{C}$, for some ordinal number α. Since $\mathfrak{C} \subseteq \mathtt{GP}(R)$, the $\mathfrak{C}$-module $M_{\beta+1}/M_\beta$ is Gorenstein projective and hence for any module $L \in \mathfrak{C}^{\diamond}$ we have

$$\widehat{\mathrm{Ext}}^0_R(M_{\beta+1}/M_\beta, L) = \underline{\mathrm{Hom}}_R(M_{\beta+1}/M_\beta, L) = 0$$

for all $\beta < \alpha$. Since the module M is Gorenstein projective as well (cf. Corollary 2.5), we may invoke Corollary 2.6 and conclude that

$$\underline{\mathrm{Hom}}_R(M, L) = \widehat{\mathrm{Ext}}^0_R(M, L) = 0.$$

As this is the case for any $L \in \mathfrak{C}^{\diamond}$, it follows that $M \in {}^{\diamond}(\mathfrak{C}^{\diamond})$. □

3. Stably finitely presented modules and orthogonality

In this section, we shall be interested in the class of stably finitely presented modules and examine the orthogonal pair which is induced by that class in the stable module category. It will turn out that this orthogonal pair enjoys certain properties which are analogous to those of cotorsion pairs with enough projective and injective objects in the category of modules (cf. [24] and [11, §4]).

If $\mathfrak{C}$ is a class of modules over a ring R and M is any module, then we denote by $\mathfrak{C}\downarrow M$ the class consisting of all pairs (C, f), where $C \in \mathfrak{C}$ and $f \in \mathrm{Hom}_R(C, M)$. The following result is based on the characterization of stably finitely presented modules provided in Lemma 1.2, coupled with the existence of a *set* of isomorphism classes of finitely presented modules.

Lemma 3.1. *Let R be a ring and consider a module M and a class $\mathfrak{C}$ of stably finitely presented modules. Then, there is a set $\Lambda = \Lambda(\mathfrak{C}, M)$ of pairs $(C', f') \in \mathfrak{C}\downarrow M$, which is such that for any $(C, f) \in \mathfrak{C}\downarrow M$ there exists a suitable pair $(C', f') \in \Lambda$ and a linear map $g \in Hom_R(C, C')$ with $[f] = [f'g] \in \underline{Hom}_R(C, M)$.*

Proof. For any finitely presented module N we consider the set $\mathrm{Hom}_R(N, M)$ and define the subset $X_N \subseteq \mathrm{Hom}_R(N, M)$, consisting of those linear maps $a : N \longrightarrow M$ for which there exist:

(i) a pair $(C, f) \in \mathfrak{C}\downarrow M$,

(ii) a module K and a projective module P,

(iii) an isomorphism $u : C \oplus K \longrightarrow N \oplus P$ and a linear map $b : P \longrightarrow M$,

such that the following diagram is commutative

$$\begin{array}{ccccccc} C & \xrightarrow{(1,0)} & C \oplus K & \xrightarrow{u} & N \oplus P & \xleftarrow{(1,0)} & N \\ {\scriptstyle f}\downarrow & & {\scriptstyle [f,0]}\downarrow & & {\scriptstyle [a,b]}\downarrow & & {\scriptstyle a}\downarrow \\ M & = & M & = & M & = & M \end{array} \tag{8}$$

For any $a \in X_N$ we choose such a 6-tuple (C, f, K, P, u, b) and label it as $(C_a, f_a, K_a, P_a, u_a, b_a)$. We note that, with the above notation, the diagram

$$\begin{array}{ccccccc} N & \xrightarrow{(1,0)} & N \oplus P_a & \xrightarrow{u_a^{-1}} & C_a \oplus K_a & \xrightarrow{[1,0]} & C_a \\ {\scriptstyle a}\downarrow & & {\scriptstyle [a,b_a]}\downarrow & & {\scriptstyle [f_a,0]}\downarrow & & {\scriptstyle f_a}\downarrow \\ M & = & M & = & M & = & M \end{array}$$

is commutative. We now consider the class of all finitely presented modules and choose a set $\mathfrak{N}$ consisting of one representative from each isomorphism class therein. We define

$$\Lambda = \bigcup_{N \in \mathfrak{N}} \{(C_a, f_a) : a \in X_N\}.$$

For any pair $(C, f) \in \mathfrak{C}{\downarrow}M$, we may use Lemma 1.2 and choose a module K and a projective module P, so that $C \oplus K \simeq N \oplus P$ for some $N \in \mathfrak{N}$. We then obtain a commutative diagram as in (8) for an isomorphism u and suitable linear maps a and b; by the very definition of X_N, we have $a \in X_N$. We note that the diagram

$$\begin{array}{ccccccccccccc} C & \xrightarrow{(1,0)} & C \oplus K & \xrightarrow{u} & N \oplus P & \xrightarrow{[1,0]} & N & \xrightarrow{(1,0)} & N \oplus P_a & \xrightarrow{u_a^{-1}} & C_a \oplus K_a & \xrightarrow{[1,0]} & C_a \\ {\scriptstyle f} \downarrow & & {\scriptstyle [f,0]} \downarrow & & {\scriptstyle [a,b]} \downarrow & & {\scriptstyle a} \downarrow & & {\scriptstyle [a,b_a]} \downarrow & & {\scriptstyle [f_a,0]} \downarrow & & {\scriptstyle f_a} \downarrow \\ M & = & M & = & M & = & M & = & M & = & M & = & M \end{array}$$

is commutative in the stable module category. In fact, all of the squares in that diagram are commutative in the ordinary module category, with the possible exception of the third one, namely of

$$\begin{array}{ccc} N \oplus P & \xrightarrow{[1,0]} & N \\ {\scriptstyle [a,b]} \downarrow & & {\scriptstyle a} \downarrow \\ M & = & M \end{array}$$

which is only stably commutative if $b \neq 0$. Hence, if we define $g : C \longrightarrow C_a$ to be composition of the six horizontal arrows in the top row of the diagram, then $[f] = [f_a g] \in \underline{\mathrm{Hom}}_R(C, M)$. Since $(C_a, f_a) \in \Lambda$, the proof of the lemma is completed. □

As an immediate consequence of the previous lemma, it follows that for any class $\mathfrak{C}$ consisting of stably finitely presented modules and for any module M, there exists a set $\mathfrak{C}_0 = \mathfrak{C}_0(M)$ of $\mathfrak{C}$-modules, which is such that

$$M \in \mathfrak{C}^{\diamond} \text{ if and only if } M \in (\mathfrak{C}_0)^{\diamond}.$$

Indeed, keeping the same notation as in the proof of Lemma 3.1, we may define $\mathfrak{C}_0$ as the set consisting of those modules $C \in \mathfrak{C}$, for which there exist a module $N \in \mathfrak{N}$ and a linear map $a \in X_N$, such that $C = C_a$. This property of $\mathfrak{C}$ is a weak (local) version of the stable analogue of an important property that a cotorsion pair may have, namely that of being cogenerated by a set of modules. As shown by Eklof and Trlifaj in [11, Theorem 10], any cotorsion pair which is cogenerated by a set of modules has enough injective (and projective) objects.

We shall now detail an explicit algebraic construction of a certain homotopy colimit, which was motivated by algebraic topology and brought into the realm of algebra by Rickard [23] and Kropholler [19]; see also [10]. This construction will turn out to be the analogue of the result by Eklof and Trlifaj mentioned above, in our setting. In order to describe the construction, we fix an Ω^{-1}-closed class $\mathfrak{C}$, which consists of stably finitely presented modules. For any module M, we shall construct a sequence of modules $(M_n)_n$ and injective linear maps

$$M_0 \xrightarrow{\iota_0} M_1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{n-1}} M_n \xrightarrow{\iota_n} \cdots,$$

in such a way that $M_0 = M$, the cokernel of $\iota_n : M_n \longrightarrow M_{n+1}$ is a hyper-$\mathfrak{C}$ module for all n and the colimit $M_\infty = \varinjlim_n M_n$ is contained in $\mathfrak{C}^{\diamond}$.

We use induction on n and begin, of course, by letting $M_0 = M$. Having constructed the modules M_k for $k = 0, 1, \ldots, n$ and the embeddings $\iota_k : M_k \longrightarrow M_{k+1}$, whose cokernels are hyper-$\mathfrak{C}$ modules for $k = 0, 1, \ldots, n-1$, we proceed with the inductive step as follows: We consider the class $\mathfrak{C}{\downarrow}M_n$ consisting of all pairs (C, f), where $C \in \mathfrak{C}$ and $f \in \mathrm{Hom}_R(C, M_n)$. In view of Lemma 3.1, we may choose a set Λ_n of pairs therein, in such a way that for any (C,f) in $\mathfrak{C}\downarrow M_n$ there exists a pair $(C',f') \in \Lambda_n$ and a suitable linear map $g \in \mathrm{Hom}_R(C,C')$ with $[f]=[f'g] \in \underline{\mathrm{Hom}}_R(C,M_n)$. We now let $C_n = \bigoplus_{(C,f)\in\Lambda_n} C$ and denote by $\eta_{(C,f)} : C \longrightarrow C_n$ the canonical embedding of C as the direct summand of C_n corresponding to any index $(C,f)\in\Lambda_n$. We also consider the linear map $f_n : C_n \longrightarrow M_n$, which is induced by the f's for all $(C,f)\in\Lambda_n$; in other words, f_n is the unique linear map for which the diagram below

$$\begin{array}{ccc} C & \stackrel{\eta_{(C,f)}}{\longrightarrow} & C_n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_n} \\ M_n & = & M_n \end{array}$$

is commutative for all $(C,f)\in\Lambda_n$. For any pair $(C,f)\in\Lambda_n$, the module C is contained in the Ω^{-1}-closed class $\mathfrak{C}$; hence, we may choose a short exact sequence

$$0 \longrightarrow C \longrightarrow P \longrightarrow D \longrightarrow 0,$$

where P is projective and $D\in\mathfrak{C}$. Taking the direct sum of these over all pairs $(C,f)\in\Lambda_n$, we obtain a short exact sequence

$$0 \longrightarrow C_n \stackrel{\jmath_n}{\longrightarrow} P_n \longrightarrow D_n \longrightarrow 0,$$

where P_n is projective and D_n is a (direct sum of modules contained in $\mathfrak{C}$ and hence a) hyper-$\mathfrak{C}$ module. We now define the module M_{n+1} as the pushout of the diagram

$$\begin{array}{ccc} C_n & \stackrel{\jmath_n}{\longrightarrow} & P_n \\ {\scriptstyle f_n}\downarrow & & \\ M_n & & \end{array}$$

In other words, M_{n+1} fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \stackrel{\jmath_n}{\longrightarrow} & P_n & \longrightarrow & D_n & \longrightarrow & 0 \\ & & {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \varphi_n} & & \| & & \\ 0 & \longrightarrow & M_n & \stackrel{\iota_n}{\longrightarrow} & M_{n+1} & \longrightarrow & D_n & \longrightarrow & 0 \end{array}$$

Since $\operatorname{coker}\iota_n \simeq D_n$ is a hyper-$\mathfrak{C}$ module, the inductive step of the construction is completed.

Having constructed as above the sequence $(M_n)_n$ and the monomorphisms ι_n, $n\geq 0$, we shall prove that the colimit $M_\infty = \varinjlim_n M_n$ is contained in $\mathfrak{C}^\diamond$. To that end, we fix a module $C\in\mathfrak{C}$, a non-negative integer n and consider a linear map $f : C \longrightarrow M_n$. In view of the defining property of the set Λ_n, there exists a pair $(C',f')\in\Lambda_n$ and a linear map $g\in\mathrm{Hom}_R(C,C')$ with $[f]=[f'g]\in\underline{\mathrm{Hom}}_R(C,M_n)$. Letting $h : C\longrightarrow C_n$ be the composition $C \stackrel{g}{\longrightarrow} C' \stackrel{\eta_{(C',f')}}{\longrightarrow} C_n$, the commutativity of the diagram

$$\begin{array}{ccccc} C & \stackrel{g}{\longrightarrow} & C' & \stackrel{\eta_{(C',f')}}{\longrightarrow} & C_n \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f_n}\downarrow \\ M_n & = & M_n & = & M_n \end{array}$$

in the stable module category shows that $[f]=[f_nh]\in\underline{\mathrm{Hom}}_R(C,M_n)$. Hence, it follows that

$$[\iota_n f] = [\iota_n f_n h] = [\varphi_n \jmath_n h] = [0] \in \underline{\mathrm{Hom}}_R(C,M_{n+1}),$$

where the latter equality is a consequence of the projectivity of the module P_n. As this is true for all $f\in\mathrm{Hom}_R(C,M_n)$, we may conclude that the additive map

$$\iota_{n*} : \underline{\mathrm{Hom}}_R(C,M_n) \longrightarrow \underline{\mathrm{Hom}}_R(C,M_{n+1}),$$

which is induced by ι_n, is the zero map for all $n \geq 0$. Since the $\mathfrak{C}$-module C is stably finitely presented and $M_\infty = \varinjlim_n M_n$, it follows that the abelian group $\underline{\mathrm{Hom}}_R(C, M_\infty)$ is identified with the colimit of the system

$$\underline{\mathrm{Hom}}_R(C, M_0) \xrightarrow{\iota_{0*}} \underline{\mathrm{Hom}}_R(C, M_1) \xrightarrow{\iota_{1*}} \cdots \xrightarrow{\iota_{n-1*}} \underline{\mathrm{Hom}}_R(C, M_n) \xrightarrow{\iota_{n*}} \cdots .$$

Therefore, the group $\underline{\mathrm{Hom}}_R(C, M_\infty)$ is trivial, as needed.

Theorem 3.2. *Let R be a ring and consider an Ω^{-1}-closed class $\mathfrak{C}$, which consists of stably finitely presented modules.*

(i) For any module M there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

where $B \in \mathfrak{C}^\diamond$ and A is a hyper-$\mathfrak{C}$ module.

(ii) Assume, in addition, that $\mathfrak{C}$ contains all projective modules. Then, for any module M there exists a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0,$$

where $B \in \mathfrak{C}^\diamond$ and A is a hyper-$\mathfrak{C}$ module.
In the special case where the class $\mathfrak{C}$ above is a subclass of $\mathtt{GP}(R)$, the module A in statements (i) and (ii) is a Gorenstein projective module contained in the double orthogonal ${}^\diamond(\mathfrak{C}^\diamond)$.

Proof. (i) For any module M we consider the sequence $(M_n)_n$ and the injective linear maps

$$M_0 \xrightarrow{\iota_0} M_1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{n-1}} M_n \xrightarrow{\iota_n} \cdots ,$$

which were constructed above; recall that $M_0 = M$, the cokernel of ι_n is a hyper-$\mathfrak{C}$ module for all n and the colimit $M_\infty = \varinjlim_n M_n$ is contained in $\mathfrak{C}^\diamond$. We regard the ι_n's as embeddings and write simply $M_\infty = \bigcup_n M_n$. Then, the cokernel of the embedding $\iota : M \hookrightarrow M_\infty$ is the module $M_\infty/M = \bigcup_n M_n/M$. Since $M_0/M = 0$ and the successive quotients $(M_{n+1}/M)/(M_n/M) \simeq M_{n+1}/M_n = \mathrm{coker}\, \iota_n$ are hyper-$\mathfrak{C}$ modules for all n, it follows that M_∞/M is a hyper-(hyper-$\mathfrak{C}$) module and hence a hyper-$\mathfrak{C}$ module as well. Therefore, the short exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} M_\infty \longrightarrow M_\infty/M \longrightarrow 0$$

satisfies the requirements in the statement to be proved. If the class $\mathfrak{C}$ consists of Gorenstein projective modules, then Corollary 2.5 implies that the hyper-$\mathfrak{C}$ module M_∞/M is Gorenstein projective, whereas Corollary 2.7 implies that $M_\infty/M \in {}^\diamond(\mathfrak{C}^\diamond)$.

(ii) We follow the (well-known by now) technique used by Salce in [24, Lemma 2.2] and consider for any given module M a projective module P and a surjective linear map $P \longrightarrow M$. Applying assertion (i) to the kernel K of that map, we deduce the existence of a short exact sequence

$$0 \longrightarrow K \longrightarrow B \longrightarrow A \longrightarrow 0,$$

where $B \in \mathfrak{C}^\diamond$ and A is a hyper-$\mathfrak{C}$ module. Then, the pushout A' of the diagram

$$\begin{array}{ccc} K & \hookrightarrow & P \\ \downarrow & & \\ B & & \end{array}$$

fits into a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & A' & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & A & = & A & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since projective modules are contained in $\mathfrak{C}$ and A' is an extension of the hyper-$\mathfrak{C}$ module A by P, it follows that A' is a hyper-$\mathfrak{C}$ module as well. Thus, the horizontal short exact sequence in the middle of the diagram satisfies the requirements in the statement to be proved. Finally, invoking Corollaries 2.5 and 2.7 as in the proof of assertion (i), we conclude that if $\mathfrak{C} \subseteq \mathtt{GP}(R)$, then A' is a Gorenstein projective module contained in ${}^{\diamond}(\mathfrak{C}^{\diamond})$.[1] □

As an application of Theorem 3.2, we shall obtain some information about the modules in the double orthogonal ${}^{\diamond}(\mathfrak{C}^{\diamond})$ of the class $\mathfrak{C}$ considered therein.

Proposition 3.3. *Let R be a ring and consider an Ω^{-1}-closed class $\mathfrak{C}$, which consists of stably finitely presented Gorenstein projective modules and contains all projective modules. Then, for any module $M \in {}^{\diamond}(\mathfrak{C}^{\diamond})$ there exists a short exact sequence*

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where P is projective and N is a direct summand of a hyper-$\mathfrak{C}$ module.

Proof. Let M be a module contained in ${}^{\diamond}(\mathfrak{C}^{\diamond})$ and consider a short exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} B \longrightarrow A \longrightarrow 0,$$

where $B \in \mathfrak{C}^{\diamond}$ and A is a Gorenstein projective hyper-$\mathfrak{C}$ module; the existence of such an exact sequence follows from Theorem 3.2. Since the abelian group $\underline{\mathrm{Hom}}_R(M, B)$ is trivial, we may factor ι as the composition $M \xrightarrow{\jmath} P \xrightarrow{g_1} B$, for a suitable projective module P and linear maps $\jmath$ and g_1. Since $g_1\jmath = \iota$ is injective, the linear map $\jmath$ is injective as well; let $N = \operatorname{coker} \jmath$ and consider the quotient map $p : P \longrightarrow N$. We also consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\jmath} & P & \xrightarrow{p} & N & \longrightarrow & 0 \\ & & \| & & \downarrow g_1 & & \downarrow f_1 & & \\ 0 & \longrightarrow & M & \xrightarrow{\iota} & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

where f_1 is the linear map obtained from g_1 by passage to the quotients. On the other hand, since the module A is Gorenstein projective and P is projective, the abelian group $\mathrm{Ext}^1_R(A, P)$ is trivial. Hence, the additive map

$$\iota^* : \mathrm{Hom}_R(B, P) \longrightarrow \mathrm{Hom}_R(M, P),$$

[1] In fact, if $\mathfrak{C} \subseteq \mathtt{GP}(R)$ then the module A is Gorenstein projective. Then, the abelian group $\mathrm{Ext}^1_R(A, P)$ is trivial and hence the vertical short exact sequence in the middle of the diagram splits, i.e. $A' \simeq A \oplus P$.

which is induced by ι, is surjective. Therefore, there exists a linear map $g_2 : B \longrightarrow P$, such that $g_2\iota = \jmath$. We now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \stackrel{\iota}{\longrightarrow} & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow g_2 & & \downarrow f_2 & & \\ 0 & \longrightarrow & M & \stackrel{\jmath}{\longrightarrow} & P & \stackrel{p}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

where f_2 is the linear map obtained from g_2 by passage to the quotients. Finally, let $g = g_2 g_1$, $f = f_2 f_1$ and consider the commutative diagram obtained by composing the previous ones

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \stackrel{\jmath}{\longrightarrow} & P & \stackrel{p}{\longrightarrow} & N & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & M & \stackrel{\jmath}{\longrightarrow} & P & \stackrel{p}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

Since $(1_P - g)\jmath = \jmath - g\jmath = 0$, we conclude that there exists a linear map $h : N \longrightarrow P$, such that $1_P - g = hp$. Then, we have

$$(1_N - f - ph)p = p - fp - php = p - fp - p(1_P - g) = p - fp - p + pg = pg - fp = 0$$

and hence $1_N - f - ph = 0$; we have therefore proved that $f_2 f_1 + ph = f + ph = 1_N$. It follows that the identity map 1_N of N factors as the composition $N \stackrel{(f_1,h)}{\longrightarrow} A \oplus P \stackrel{[f_2,p]}{\longrightarrow} N$ and hence N is a direct summand of $A \oplus P$. Since the projective module P is contained in $\mathfrak{C}$, the direct sum $A \oplus P$ is a hyper-$\mathfrak{C}$ module and the proof is complete. $\square$

Remarks 3.4. (i) Let $\mathfrak{C}$ be an Ω^{-1}-closed class, which consists of stably finitely presented Gorenstein projective modules and contains all projective modules. The double orthogonal ${}^{\diamond}(\mathfrak{C}^{\diamond})$ is closed under direct summands and contains all hyper-$\mathfrak{C}$ modules (cf. Corollary 2.7). In particular, for any module $M \in {}^{\diamond}(\mathfrak{C}^{\diamond})$ the module N in the statement of Proposition 3.3 is contained in ${}^{\diamond}(\mathfrak{C}^{\diamond})$ as well. Hence, Proposition 3.3 implies that the class ${}^{\diamond}(\mathfrak{C}^{\diamond})$ is Ω^{-1}-closed.

(ii) A careful examination of the proofs that were provided above for Theorem 3.2(ii) and Proposition 3.3 shows that in both of these results the hypothesis that $\mathfrak{C}$ contains all projective modules may be weakened; in fact, it only suffices to assume therein that the regular module is contained in $\mathfrak{C}$ (or even that R is a hyper-$\mathfrak{C}$ module).

4. Completely finitary Gorenstein projective modules

In this final section, we specialize the discussion to the class of completely finitary Gorenstein projective modules. Taking into account a description of the modules in the right orthogonal of that class in terms of complete cohomology, the results of the previous section will lead to the proof of our main result, namely of the Theorem stated in the Introduction.

A class $\mathfrak{C}$ of modules is called $\Omega^{\pm 1}$-closed if it is both Ω^1- and Ω^{-1}-closed. The property of the modules in the double orthogonal ${}^{\diamond}(\mathfrak{C}^{\diamond})$ presented in Proposition 3.3 is characteristic of the modules in that class, in the special case where the class $\mathfrak{C}$ therein is $\Omega^{\pm 1}$-closed. The following result is a stable analogue of [15, Corollary 3.2.4].

Proposition 4.1. *Let R be a ring and consider an $\Omega^{\pm 1}$-closed class $\mathfrak{C}$ consisting of stably finitely presented Gorenstein projective modules and containing all projective modules. Then, the following conditions are equivalent for a module M:*

(i) $M \in {}^{\diamond}(\mathfrak{C}^{\diamond})$,

(ii) there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where P is projective and N is a direct summand of a hyper-$\mathfrak{C}$ module, and

(iii) M is a direct summand of a hyper-$\mathfrak{C}$ module.

Proof. (i)→(ii): This is proved in Proposition 3.3.

(ii)→(iii): Let M be a module for which there exists a short exact sequence as in (ii). Then, $N \oplus N'$ is a hyper-$\mathfrak{C}$ module for a suitable module N'. We consider a short exact sequence

$$0 \longrightarrow M' \longrightarrow P' \longrightarrow N' \longrightarrow 0,$$

where P' is a projective module, and let

$$0 \longrightarrow M \oplus M' \longrightarrow P \oplus P' \longrightarrow N \oplus N' \longrightarrow 0$$

be the direct sum of the two exact sequences. Since the class $\mathfrak{C}$ is Ω^1-closed, we know that the class hyper-$\mathfrak{C}$ is also Ω^1-closed (cf. Remark 2.3(iii)); hence, there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow Q \longrightarrow N \oplus N' \longrightarrow 0,$$

where Q is projective and K is a hyper-$\mathfrak{C}$ module. As the projective module $P \oplus P'$ is contained in $\mathfrak{C}$, it follows that $K \oplus P \oplus P'$ is a hyper-$\mathfrak{C}$ module as well. This completes the proof, since M is a direct summand of the latter module, in view of Schanuel's lemma.

(iii)→(i): Since the double orthogonal ${}^{\diamond}(\mathfrak{C}^{\diamond})$ is closed under direct summands, this follows from Corollary 2.7. □

Remark 4.2. Let $\mathfrak{C}$ be an $\Omega^{\pm 1}$-closed class, consisting of stably finitely presented Gorenstein projective modules and containing all projective modules. Then, Proposition 4.1 implies that the class ${}^{\diamond}(\mathfrak{C}^{\diamond})$ is $\Omega^{\pm 1}$-closed as well.

Let $\mathfrak{C}$ be an $\Omega^{\pm 1}$-closed class and consider a module $C \in \mathfrak{C}$. Then, using an inductive argument, we may construct a doubly infinite acyclic complex of projective modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots,$$

whose syzygy modules are all contained in $\mathfrak{C}$ and whose 0-th syzygy module is C. Using this observation, we shall now prove that the right orthogonal of an $\Omega^{\pm 1}$-closed class consisting of Gorenstein projective modules admits a rigid description, as far as the complete cohomology functors are concerned.

Lemma 4.3. *Let R be a ring and consider an $\Omega^{\pm 1}$-closed class $\mathfrak{C}$, which consists of Gorenstein projective modules. Then, the following conditions are equivalent for a module L:*

(i) $L \in \mathfrak{C}^{\diamond}$ and

(ii) $\widehat{\mathrm{Ext}}^i_R(C, L) = 0$ for all modules $C \in \mathfrak{C}$ and all integers $i \in \mathbb{Z}$.

Proof. The implication (ii)→(i) follows since $\mathfrak{C}$ is a subclass of $\mathtt{GP}(R)$ and hence $\widehat{\mathrm{Ext}}^0_R(C, \_) = \underline{\mathrm{Hom}}_R(C, \_)$ for any module $C \in \mathfrak{C}$.

In order to prove that (i)→(ii), assume that (i) holds and consider a $\mathfrak{C}$-module C. As we noted above, there exists a doubly infinite acyclic complex of projective modules, with i-th syzygy module C_i contained in $\mathfrak{C}$ for all $i \in \mathbb{Z}$ and $C_0 = C$. Since $\mathfrak{C}$ is a subclass of $\mathtt{GP}(R)$, the module C_i is Gorenstein projective for all i. Hence, using dimension shifting and assumption (i), it follows that

$$\widehat{\mathrm{Ext}}^i_R(C, L) = \widehat{\mathrm{Ext}}^i_R(C_0, L) = \widehat{\mathrm{Ext}}^0_R(C_i, L) = \underline{\mathrm{Hom}}_R(C_i, L) = 0$$

for all $i \in \mathbb{Z}$, as needed. □

Proposition 4.4. *Let R be a ring and consider an $\Omega^{\pm 1}$-closed class $\mathfrak{C}$, which consists of stably finitely presented Gorenstein projective modules.*

(i) For any module M there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

where A is a Gorenstein projective hyper-$\mathfrak{C}$ module and B is such that $\mathrm{Ext}^1_R(C,B)=0$ for all modules $C \in \mathfrak{C}$.

(ii) Assume, in addition, that $\mathfrak{C}$ contains all projective modules. Then, for any module M there exists a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0,$$

where A is a Gorenstein projective hyper-$\mathfrak{C}$ module and B is such that $\mathrm{Ext}^1_R(C,B)=0$ for all modules $C \in \mathfrak{C}$.

Proof. Invoking Theorem 3.2, we can find for any module M short exact sequences as in the statement of the proposition, where A is a Gorenstein projective hyper-$\mathfrak{C}$ module and $B \in \mathfrak{C}^{\diamond}$. Then, Lemma 4.3 implies that the abelian group $\mathrm{Ext}^1_R(C,B) = \widehat{\mathrm{Ext}}^1_R(C,B)$ is trivial for all modules $C \in \mathfrak{C}$. □

Remarks 4.5. (i) The class $\mathtt{CF}(R) \cap \mathtt{GP}(R)$ of all completely finitary Gorenstein projective modules provides an example of a class that satisfies the hypotheses of Proposition 4.4. Indeed, $\mathtt{CF}(R) \cap \mathtt{GP}(R)$ consists of stably finitely presented Gorenstein projective modules and contains all projective modules. Furthermore, the class $\mathtt{CF}(R) \cap \mathtt{GP}(R)$ is $\Omega^{\pm 1}$-closed, since all syzygy modules of an acyclic complex of projective modules are completely finitary, provided that one of them is.

(ii) Any $\Omega^{\pm 1}$-closed class $\mathfrak{C}$, which consists of stably finitely presented Gorenstein projective modules, is contained in $\mathtt{CF}(R) \cap \mathtt{GP}(R)$.
In order to verify this, let $\mathfrak{C}$ be an $\Omega^{\pm 1}$-closed class consisting of stably finitely presented Gorenstein projective modules and consider a $\mathfrak{C}$-module C. We also consider a doubly infinite acyclic chain complex of projective modules, whose i-th syzygy module C_i is contained in $\mathfrak{C}$ for all i and whose 0-th syzygy module C_0 equals C. Using dimension shifting, it follows that the complete cohomology functors $\widehat{\mathrm{Ext}}^i(C,\_) = \widehat{\mathrm{Ext}}^i(C_0,\_)$ and $\widehat{\mathrm{Ext}}^0(C_i,\_)$ are naturally isomorphic for all $i \in \mathbb{Z}$. Our assumption that $\mathfrak{C} \subseteq \mathtt{SFP}(R) \cap \mathtt{GP}(R)$ implies that the module C_i is stably finitely presented and Gorenstein projective; hence, the functor $\widehat{\mathrm{Ext}}^0(C_i,\_) = \underline{\mathrm{Hom}}_R(C_i,\_)$ commutes with filtered colimits for all i. It follows readily that C is completely finitary, as needed.

We shall conclude with the following corollary, some parts of which constitute the Theorem stated in the Introduction.

Corollary 4.6. *Let R be a ring.*

(i) For any module M there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0,$$

where A is a hyperfinite extension of completely finitary Gorenstein projective modules (and hence A is Gorenstein projective) and B is such that $\mathrm{Ext}^1_R(C,B)=0$ for all completely finitary Gorenstein projective modules C.

(ii) For any module M there exists a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0,$$

where A is a hyperfinite extension of completely finitary Gorenstein projective modules (and hence A is Gorenstein projective) and B is such that $Ext^1_R(C,B)=0$ for all completely finitary Gorenstein projective modules C.

Proof. This follows from Proposition 4.4, by letting $\mathfrak{C} = \mathtt{CF}(R) \cap \mathtt{GP}(R)$ therein. □

Remarks 4.7. (i) Let $\mathfrak{C} = \mathtt{CF}(R) \cap \mathtt{GP}(R)$ and consider the double orthogonal $\overline{\mathfrak{C}} = {}^{\diamond}(\mathfrak{C}^{\diamond})$; in view of Proposition 4.1, the class $\overline{\mathfrak{C}}$ consists of all direct summands of hyperfinite extensions of completely finitary Gorenstein projective modules. Since the class of Gorenstein projective modules is closed under hyperfinite extensions (cf. Corollary 2.5) and direct summands [17], it follows that $\overline{\mathfrak{C}}$ is a subclass of $\mathtt{GP}(R)$. The class $\overline{\mathfrak{C}}$ is precovering; in fact, for any module M the short exact sequence of Corollary 4.6(ii) is a special $\overline{\mathfrak{C}}$-precover.
Indeed, keeping the same notation as in Corollary 4.6(ii), we have $A \in \overline{\mathfrak{C}}$. On the other hand, Auslander's lemma implies that the functor $\mathrm{Ext}^1_R(\_, B)$ vanishes on all hyperfinite extensions of completely finitary Gorenstein projective modules and hence on all direct summands of these (i.e. on all $\overline{\mathfrak{C}}$-modules).

(ii) Having the existence of (special) Gorenstein projective precovers in our mind, it would be of some interest to examine rings over which *any* Gorenstein projective module may be expressed as a hyperfinite extension of completely finitary Gorenstein projective modules; in the notation of (i) above, we would then have an equality $\overline{\mathfrak{C}} = \mathtt{GP}(R)$. As an example, we note that Beligiannis has characterized in [6] the Artin algebras over which any Gorenstein projective module may be decomposed into the direct sum of finitely generated modules (which are then necessarily of type FP_∞ and Gorenstein projective). Of course, the class $\mathtt{GP}(R)$ is already known to be precovering when R is an Artin algebra; this case is covered by Jorgensen's result [18] about Noetherian algebras over a field that admit a dualizing complex.

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