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ON PURE ACYCLIC COMPLEXES

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Abstract. In this paper, we study the pure acyclic complexes of modules. We obtain several characterizations of these complexes, extending results that are known for the pure acyclic complexes of flat modules. In particular, we extend Neeman's characterization [14] of the pure acyclic complexes of flat modules in terms of complexes of projective modules, by considering more generally complexes of pure projective modules. As a consequence, we obtain Simson's result [18] on the pure projectivity of pure periodic modules.

Contents

0. Introduction

Flat modules were originally introduced in the realm of algebraic geometry [16], in order to properly formulate and study the notion of a family of varieties that depend continuously on a parameter. This concept was also applied in the non-commutative setting; it was soon realized that the properties of flat modules reflect properties of the ideal structure of the coefficient ring. For example, Bass proved in [2] that a ring is left perfect (i.e. it satisfies the descending chain condition on principal right ideals) if and only if any flat module is projective, whereas Chase proved in [5] that a ring is right coherent (i.e. any finitely generated right ideal is finitely presented as a right module) if and only if all direct products of flat modules are flat. The description of flat modules as the direct limits of finitely generated projective modules, which was obtained independently by Lazard [12] and Govorov [9], was proved to be very useful in the study of the relation between projectivity and flatness. It turns out that the finiteness of the projective dimension of flat modules is related to both set theoretic and geometric properties of the ambient ring: All flat modules have finite projective dimension in the case where the ring has cardinality $\aleph_n$ for some $n \in \mathbb{N}$ (this result is due to Simson [17]) or the ring is commutative Noetherian of finite Krull dimension (this result is due to Raynaud and Gruson [15]). It is worth mentioning that Raynaud and Gruson obtained in [loc.cit.] a necessary and sufficient condition for a flat module to be projective, in terms of the notion of Mittag-Leffler modules. On the other hand, flat modules have been useful in the study of duality phenomena, as they play an intermediary role between projective and injective modules in various duality schemes. This is examplified by Pontryagin duality: Lambek has proved in [11] that a module is flat if and only if its character module is injective. Flat modules also play a central role in

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Grothendieck's theory of duality in algebraic geometry, via the concept of dualizing complexes [10]; see also [14, §2].

Some more recent advances in the theory of flat modules suggest that these modules possess several intriguing and unexpected properties. As shown by Benson and Goodearl in [3], a flat module M is projective, if it fits into an exact sequence of the form

$$(1) \qquad 0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is projective. This result has been generalized in two directions (which are, in some sense, perpendicular to each other). The first one of these is due to Simson and involves the notion of purity. This notion is closely related to flatness and has been very successfully used in homological algebra; it suffices to mention that a proof of the flat cover conjecture may be obtained by studying filtrations of a module by pure submodules [4]. Purity is the basis of a relative homological theory, which leads to such concepts as pure projective modules, pure resolutions by pure projective modules, pure projective dimension, etc. Simson has proved in [18] that a module M is pure projective, if it fits into a pure exact sequence as in (1) above, where P is pure projective. Since the flatness of M implies that the short exact sequence (1) is pure, whereas any module which is both flat and pure projective is necessarily projective, Simson's result is indeed a generalization of the above result by Benson and Goodearl.

On the other hand, Neeman has studied in [14] the embedding of the homotopy category of projective modules into that of flat modules. Using the theory of well-generated triangulated categories and Brown representability, he proved that this embedding has a right adjoint. He showed that the kernel of that adjoint consists of the pure acyclic complexes of flat modules, i.e. of those acyclic complexes of flat modules, whose syzygy modules are flat. These complexes have been studied by many authors under several names; they are called categorically flat in [1], flat in [8], acyclic semi-flat in [6] and, finally, pure acyclic in [13]. It follows from Neeman's characterization that a complex of flat modules F is pure acyclic if and only if any chain map from a complex of projective modules to F is null-homotopic. In particular, a pure acyclic complex of projective modules must necessarily be contractible, i.e. all of its syzygy modules must be projective. Since the short exact sequence (1) induces, in the case where the modules P and M therein are projective and flat respectively, a doubly infinite pure acyclic complex of projective modules, one may easily recover the result by Benson and Goodearl from Neeman's description.

In this paper, we extend Simson's result on the pure projectivity of pure periodic modules and Neeman's characterization of the pure acyclic complexes of flat modules, by relating the pure acyclic complexes (of not necessarily flat modules) to the complexes of pure projective modules. We present a long list of characterizations of the pure acyclic complexes, which are based on certain well-known conditions that characterize the purity of a short exact sequence. This list of conditions generalizes and complements the corresponding list of conditions for the pure acyclic complexes of flat modules, obtained by Enochs and Garcia Rojas [8], Neeman [14] and Christensen and Holm [6]. In particular, we prove the following result.

Theorem. *A complex F is pure acyclic if and only if any chain map from a complex of pure projective modules to F is null-homotopic.*

Our proof of the result above provides also a proof of Neeman's characterization of the pure acyclic complexes of flat modules in terms of the complexes of projective modules, that avoids any use of the theory of triangulated categories. At the same time, we should point out that our arguments are heavily influenced by Neeman's.

Here is a brief outline of the contents of the paper: In Section 1, we examine the properties of the pure acyclic complexes that are related to tensor products by modules and complexes. In the following section, we examine the properties of these complexes with respect to left or right bounded complexes of finitely presented modules. In Section 3, we extend Neeman's characterization of the pure acyclic complexes of flat modules in terms of the complexes of projective modules and prove the Theorem stated above, relating the pure acyclic complexes to the complexes of pure projective modules.

Notations and terminology. Unless otherwise specified, all modules considered in this paper are left modules over a fixed associative unital ring R. We denote by R^o the opposite ring of R. If X, Y and Z are three modules, then we identify the abelian group $\mathrm{Hom}_R(X, Y \oplus Z)$ with the direct sum $\mathrm{Hom}_R(X, Y) \oplus \mathrm{Hom}_R(X, Z)$; an element $(f, g) \in \mathrm{Hom}_R(X, Y) \oplus \mathrm{Hom}_R(X, Z)$ is thus identified with the map $X \longrightarrow Y \oplus Z$, which is given by $x \mapsto (f(x), g(x))$, $x \in X$. There is an analogous identification $\mathrm{Hom}_R(X \oplus Y, Z) = \mathrm{Hom}_R(X, Z) \oplus \mathrm{Hom}_R(Y, Z)$; if $f \in \mathrm{Hom}_R(X, Z)$ and $g \in \mathrm{Hom}_R(Y, Z)$, then we denote by $[f, g] : X \oplus Y \longrightarrow Z$ the corresponding linear map, which is itself given by $(x, y) \mapsto f(x) + g(y)$, $(x, y) \in X \oplus Y$.

All complexes are indexed homologically. Any module M is regarded as a complex consisting of M in degree 0 (and zeroes elsewhere). Let X be a complex with differential ∂^X and fix an integer n. Then, we denote by $X[n]$ the complex consisting of X_{i-n} in degree i with differential $(-1)^n \partial^X$. The truncation $X^{\leq n}$ is the subcomplex of X consisting of X_i (resp. 0) in degrees $i \leq n$ (resp. $i > n$). If the complex X is acyclic, then its n-th syzygy module is defined to be the module of n-cycles, i.e. the kernel $Z_n X$ of the differential $\partial^X : X_n \longrightarrow X_{n-1}$.

1. The relation to tensor products

In this section, we shall present several more or less known conditions that characterize the pure acyclic complexes, involving the preservation of exactness upon tensoring the complex with modules or complexes. In the case of complexes of flat modules, these conditions are due to Enochs and Garcia Rojas [8], Neeman [14] and Christensen and Holm [6].

Following Cohn [7], a short exact sequence of modules

$$0 \longrightarrow M' \stackrel{\iota}{\longrightarrow} M \stackrel{p}{\longrightarrow} M'' \longrightarrow 0 \tag{2}$$

is called pure if it remains exact upon tensoring with any right module N; in that case, we also say that ι (resp. p) is a pure monomorphism (resp. a pure epimorphism). It is clear that any short exact sequence as above is pure if the module M'' is flat. In fact, if M is flat, then the short exact sequence (2) is pure only if M'' is flat; in that case, M' is flat as well.

The notion of purity of a short exact sequence admits several equivalent formulations. One of these involves the character (or Pontryagin dual) modules. We recall that the Pontryagin duality functors D from the category of left (resp. right) modules to the category of right (resp. left) modules are defined by $M \mapsto \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. Since D is a contravariant exact functor, any short exact sequence of modules (2) induces a short exact sequence of right modules

$$0 \longrightarrow DM'' \stackrel{Dp}{\longrightarrow} DM \stackrel{D\iota}{\longrightarrow} DM' \longrightarrow 0.$$

It turns out that the short exact sequence (2) is pure if and only if the associated short exact sequence of Pontryagin duals is split (cf. [15, §II.1.1.1]).

In particular, we may define the functor D on abelian groups. Then, an abelian group A is trivial if and only if $DA = 0$; this follows since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator of the category of abelian groups. If M is a right module and N is a left module, then the standard Hom-tensor duality induces a natural identification $D(M \otimes_R N) = \mathrm{Hom}_{R^o}(M, DN) = \mathrm{Hom}_R(N, DM)$.

We shall be interested in the pure acyclic complexes of modules; by definition, these are the acyclic complexes F, which are such that the short exact sequences of modules

$$0 \longrightarrow Z_n F \longrightarrow F_n \longrightarrow Z_{n-1}F \longrightarrow 0$$

are pure for all n. It follows that an acyclic complex of flat modules is pure acyclic if and only if all of its syzygy modules are flat.

Proposition 1.1. *The following conditions are equivalent for a complex F:*

(i) F is pure acyclic,

(ii) DF is contractible,

(iii) $X \otimes_R F$ is acyclic for any complex of right modules X,

(iv) $M \otimes_R F$ is acyclic for any right module M,

(v) $M \otimes_R F$ is acyclic, where $M = R \oplus (\bigoplus_n DZ_nF)$ is the direct sum of the right regular module and the Pontryagin duals $(DZ_nF)_n$ of the cycle modules $(Z_nF)_n$ of F,

(vi) F is acyclic and the Pontryagin dual DZ_nF of its syzygy module Z_nF induces an acyclic complex $DZ_nF \otimes_R F$ for all n,

(vii) F is acyclic and $X \otimes_R F$ is acyclic for any acyclic complex of right modules X,

(viii) $X \otimes_R F$ is acyclic for any right bounded complex of right modules X, which either consists of flat modules or else is acyclic.

Proof. We shall prove that (i)→(ii)→(iii)→(iv)→(v)→(vi)→(i) and (iii)→(vii)→(viii)→(iv).

(i)→(ii): Since F is acyclic, the complex DF is also acyclic. Since the complex F is actually pure acyclic, the short exact sequences

$$0 \longrightarrow Z_nF \longrightarrow F_n \longrightarrow Z_{n-1}F \longrightarrow 0 \tag{3}$$

are pure and hence they induce split short exact sequences of right modules

$$0 \longrightarrow DZ_{n-1}F \longrightarrow DF_n \longrightarrow DZ_nF \longrightarrow 0 \tag{4}$$

for all n. As the complex DF is built up by splicing these, it follows that it is contractible.

(ii)→(iii): It suffices to prove that the complex $D(X \otimes_R F) = \mathrm{Hom}_{R^o}(X, DF)$ is acyclic. In fact, the contractibility of DF implies that the complex $\mathrm{Hom}_{R^o}(X, DF)$ is even contractible.

(iii)→(iv): This is obvious.

(iv)→(v): This is obvious.

(v)→(vi): In view of the identification $R \otimes_R F = F$, this is obvious as well.

(vi)→(i): In order to prove that the short exact sequences (3) are pure, it suffices to show that the short exact sequences (4) are split. To that end, we fix an integer n and note that the identification

$$D(DZ_nF \otimes_R F) = \mathrm{Hom}_{R^o}(DZ_nF, DF)$$

implies that the complex $\mathrm{Hom}_{R^o}(DZ_nF, DF)$ is acyclic. In particular, the embedding of DZ_nF into DF_{n+1} must factor through DF_n, providing a splitting of (4), as needed.

(iii)→(vii): This is obvious, since the complex $R[0] \otimes_R F = R \otimes_R F$ is identified with F.

(vii)→(viii): It suffices to prove that the acyclicity of F implies that $X \otimes_R F$ is acyclic for any right bounded complex of flat right modules X. This claim is proved in the next lemma.

Lemma 1.2. *If F is an acyclic complex of modules and X is a right bounded complex of flat right modules, then the complex $X \otimes_R F$ is acyclic.*

Proof. It suffices to prove that the complex $D(X \otimes_R F) = \mathrm{Hom}_R(F, DX)$ is acyclic. Since X is a right bounded complex of flat right modules, the complex DX is left bounded and consists of injective modules (cf. [11]). Hence, the contractibility of the Hom-complex $\mathrm{Hom}_R(F, DX)$ is a restatement of the following well-known property: Any chain map from an acyclic complex to a left bounded complex of injective modules is null-homotopic. □

Proof of Proposition 1.1. (cont.) (viii)→(iv): Let M be a right module and fix a flat resolution $F' \longrightarrow M \longrightarrow 0$ of it. We also consider the augmented complex X which is associated with the given resolution and note that there is a degree-wise split short exact sequence of complexes of right modules

$$0 \longrightarrow M[-1] \longrightarrow X \longrightarrow F' \longrightarrow 0.$$

Then, there is an induced short exact sequence of complexes

$$0 \longrightarrow (M \otimes_R F)[-1] \longrightarrow X \otimes_R F \longrightarrow F' \otimes_R F \longrightarrow 0.$$

Since the complexes F' and X are right bounded, F' is a complex of flat right modules and X is acyclic, our hypothesis implies that the complexes $F' \otimes_R F$ and $X \otimes_R F$ are both acyclic. It follows that the translate $(M \otimes_R F)[-1]$ of $M \otimes_R F$ is acyclic as well. □

2. The relation to finitely presented modules

In this section, we present certain characterizations of the pure acyclic complexes, in terms of (complexes of) finitely presented modules. In the special case of pure acyclic complexes of flat modules, these characterizations are due to Enochs and Garcia Rojas [8], Neeman [14] and Christensen and Holm [6].

The results that are obtained in this section are based on (and, at the same time, generalize) a well-known criterion of purity that involves finitely presented modules. It is known that a short exact sequence of modules

$$0 \longrightarrow M' \xrightarrow{\iota} M \xrightarrow{p} M'' \longrightarrow 0$$

is pure if and only if the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(C, M') \xrightarrow{\iota_*} \mathrm{Hom}_R(C, M) \xrightarrow{p_*} \mathrm{Hom}_R(C, M'') \longrightarrow 0 \tag{5}$$

is exact for any finitely presented module C (cf. [15, §II.1.1.1]). Of course, this is the case if and only if the additive map

$$p_* : \mathrm{Hom}_R(C, M) \longrightarrow \mathrm{Hom}_R(C, M''),$$

which is induced by the linear map p, is surjective for any finitely presented module C. Then, it is easily seen that the sequence (5) is exact (i.e. the additive map p_* is surjective) in the more general case where C is a direct summand of a direct sum of finitely presented modules. In fact, it is known that the direct summands of the direct sums of finitely presented modules are precisely the modules that induce such a short exact sequence of abelian groups for any pure exact sequence of modules (cf. [15, §II.1.1.2]); these modules are called pure projective. Any module M admits a pure epimorphism from a pure projective module; for example, if M is expressed as the colimit $\varinjlim_i M_i$ of a directed system of finitely presented modules, then the canonical map $\bigoplus_i M_i \longrightarrow M$ is a pure epimorphism (cf. [15, §II.1.1.3]).

We begin with the following pure version of the horseshoe lemma.

Lemma 2.1. *(pure horseshoe lemma) Let*

$$0 \longrightarrow M' \xrightarrow{\iota} M \xrightarrow{p} M'' \longrightarrow 0$$

be a pure exact sequence of modules and consider two pure exact sequences

$$0 \longrightarrow N' \longrightarrow P' \xrightarrow{\pi'} M' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N'' \longrightarrow P'' \xrightarrow{\pi''} M'' \longrightarrow 0,$$

where P'' is pure projective. Then, there exists a pure exact sequence

$$0 \longrightarrow N \longrightarrow P' \oplus P'' \xrightarrow{\pi} M \longrightarrow 0,$$

which fits into a commutative diagram with pure exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \longrightarrow & P' \oplus P'' & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & {\scriptstyle \pi'} \downarrow & & {\scriptstyle \pi} \downarrow & & {\scriptstyle \pi''} \downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{\iota} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(In the diagram above, the second row involves the natural embedding of P' into the direct sum and the natural projection of the direct sum onto P''.)

Proof. In view of our assumptions, the additive map

$$p_* : \mathrm{Hom}_R(P'', M) \longrightarrow \mathrm{Hom}_R(P'', M''),$$

which is induced by p, is surjective. Therefore, there exists a linear map $f : P'' \longrightarrow M$, such that $pf = \pi''$. Then, the linear map $\pi = [\iota\pi', f] : P' \oplus P'' \longrightarrow M$ fits into the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \longrightarrow & P' \oplus P'' & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & {\scriptstyle \pi'} \downarrow & & {\scriptstyle \pi} \downarrow & & {\scriptstyle \pi''} \downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{\iota} & M & \xrightarrow{p} & M'' & \longrightarrow & 0
\end{array}$$

We now let $N = \ker \pi$ and use the snake lemma in order to obtain a diagram with exact rows and columns as in the statement. Tensoring with a right module and using again the snake lemma, we conclude that the horizontal short exact sequence in the top and the vertical short exact sequence in the middle of that diagram are also pure, as needed. □

We recall that a small category I is called filtered if it satisfies the following two conditions:

(filt-1) For any two objects i, j of I there exists an object k, which is such that the Hom-sets $\mathrm{Hom}_I(i, k)$ and $\mathrm{Hom}_I(j, k)$ are both non-empty.

(filt-2) For any two objects i, j of I and any two parallel morphisms $f, g \in \mathrm{Hom}_I(i, j)$, there exists an object k and a morphism $h \in \mathrm{Hom}_I(j, k)$, such that $hf = hg \in \mathrm{Hom}_I(i, k)$.

Examples of filtered categories are provided by directed ordered sets. The colimit of functors which are defined on a filtered category with values in the category of modules or the category of chain complexes of modules (such colimits are called filtered colimits) is known to be itself an exact functor.

Since tensor products commute with filtered colimits and a filtered colimit of injective maps is also injective, it follows that a filtered colimit of pure exact sequences is pure exact as well.

In particular, a filtered colimit of split short exact sequences is pure exact. In fact, it is known that a short exact sequence of modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is pure if and only if it is the colimit of a direct system of split short exact sequences

$$0 \longrightarrow C_i' \longrightarrow C_i \longrightarrow C_i'' \longrightarrow 0,$$

where the modules C_i' and C_i'' (and hence C_i as well) are finitely presented for all i (cf. [15, §II.1.1.1]). We shall now extend that characterization of purity to the pure acyclic complexes.

Proposition 2.2. *The following conditions are equivalent for a complex F:*

(i) F is pure acyclic,

(ii) $Hom_R(C,F)$ is acyclic for any finitely presented module C,

(iii) any chain map from a right bounded complex of finitely presented modules to F is null-homotopic,

(iv) F is acyclic and for any chain map $f: Y \longrightarrow F$, which consists of pure epimorphisms in each degree, and for any right bounded complex of finitely presented modules C, any chain map $C \longrightarrow F$ can be factored through f,

(v) any chain map from a right bounded complex of finitely presented modules to F can be factored through a right bounded contractible complex of finitely presented modules,

(vi) any chain map from a bounded complex of finitely presented modules to F can be factored through a bounded contractible complex of finitely presented modules,

(vii) F is a filtered colimit of bounded contractible complexes of finitely presented modules,

(viii) F is a filtered colimit of contractible complexes.

If F is a complex of flat modules, then these conditions are also equivalent to:

(flat-iv) F is acyclic and for any surjective chain map $f: Y \longrightarrow F$ and for any right bounded complex of finitely presented modules C, any chain map $C \longrightarrow F$ can be factored through f,

(flat-v) any chain map from a right bounded complex of finitely presented modules to F can be factored through a right bounded contractible complex of finitely generated free modules,

(flat-vi) any chain map from a bounded complex of finitely presented modules to F can be factored through a bounded contractible complex of finitely generated projective modules,

(flat-vii) F is a filtered colimit of bounded contractible complexes of finitely generated projective modules, and

(flat-viii) F is a filtered colimit of contractible complexes of projective modules.

Proof. (i)→(ii): Let C be a finitely presented module. Since F is pure acyclic, there are pure short exact sequences

$$0 \longrightarrow Z_n F \longrightarrow F_n \longrightarrow Z_{n-1} F \longrightarrow 0,$$

which induce short exact sequences of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(C, Z_n F) \longrightarrow \mathrm{Hom}_R(C, F_n) \longrightarrow \mathrm{Hom}_R(C, Z_{n-1} F) \longrightarrow 0$$

for all n. As the complex $\mathrm{Hom}_R(C,F)$ is built up by splicing them, it follows that it is acyclic.

(ii)→(iii): This is an immediate consequence of the following lemma.

Lemma 2.3. *Let X, Y be two chain complexes and assume that:*

(i) the complex $Hom_R(X_n, Y)$ is acyclic for all n and

(ii) $Hom_R(X_n, Y_n)$ is the trivial group for all $n \ll 0$.

Then, any chain map $f: X \longrightarrow Y$ is null-homotopic.

Proof. For any chain map $f : X \longrightarrow Y$ we shall construct linear maps $\Sigma_n : X_n \longrightarrow Y_{n+1}$, such that $f_n = \partial^Y \Sigma_n + \Sigma_{n-1}\partial^X$ for all n. We define $\Sigma_n = 0$ for all $n \ll 0$ and proceed by (ascending) induction on n. We assume therefore that n is an integer and the construction of the Σ_i's has been performed for all $i < n$. Then, the linear map $f_n - \Sigma_{n-1}\partial^X : X_n \longrightarrow Y_n$ is a n-cycle of the complex $\mathrm{Hom}_R(X_n, Y)$, since

$$\begin{array}{rcl} \partial^Y(f_n - \Sigma_{n-1}\partial^X) & = & \partial^Y f_n - \partial^Y \Sigma_{n-1}\partial^X \\ & = & \partial^Y f_n - (f_{n-1} - \Sigma_{n-2}\partial^X)\partial^X \\ & = & \partial^Y f_n - f_{n-1}\partial^X + \Sigma_{n-2}\partial^X\partial^X \\ & = & \partial^Y f_n - f_{n-1}\partial^X \\ & = & 0. \end{array}$$

In view of (i), there exists a linear map $\Sigma_n : X_n \longrightarrow Y_{n+1}$, such that $f_n - \Sigma_{n-1}\partial^X = \partial^Y\Sigma_n$, and hence the inductive step is complete. □

Proof of Proposition 2.2. (cont.) (iii)→(iv): Any n-cycle of F can be detected by a chain map $R[n] \longrightarrow F$, where $R[n]$ is the complex consisting of R in degree n and zeroes elsewhere, as the image of $1 \in R$. Hence, condition (iii) is easily seen to imply that F is acyclic. We now let C be a right bounded complex of finitely presented modules and consider a chain map $g : C \longrightarrow F$. In view of our assumption, there exists a chain homotopy Σ between g and the zero map, i.e. a homogeneous map $\Sigma : C \longrightarrow F$ of degree one, such that $g = \partial^F\Sigma + \Sigma\partial^C$. If $f : Y \longrightarrow F$ is a chain map consisting of pure epimorphisms in each degree, then there exists a homogeneous map $S : C \longrightarrow Y$ of degree 1, such that $\Sigma = fS$. Then, $h = \partial^Y S + S\partial^C : C \longrightarrow Y$ is a chain map (since $\partial^Y h = \partial^Y S \partial^C = h\partial^C$), such that

$$fh = f(\partial^Y S + S\partial^C) = f\partial^Y S + fS\partial^C = \partial^F fS + fS\partial^C = \partial^F\Sigma + \Sigma\partial^C = g.$$

In the above chain of equalities, the third one follows since f is a chain map.

(iv)→(v): Since the complex F is acyclic, there exists a contractible complex P, all of whose syzygy modules are direct sums of finitely presented modules (so that the chain modules of P are direct sums of finitely presented modules as well), and a chain map $\pi : P \longrightarrow F$, which is a pure epimorphism in each degree. We may construct such a pair (P, π) as follows: If Q_n is a direct sum of finitely presented modules that maps via a pure epimorphism onto the syzygy module Z_nF of F for all n, then we may invoke the pure horseshoe lemma (Lemma 2.1), in order to find a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_{n+1} & \longrightarrow & Q_{n+1} \oplus Q_n & \longrightarrow & Q_n & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle \pi_n} \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_{n+1}F & \longrightarrow & F_n & \longrightarrow & Z_nF & \longrightarrow & 0 \end{array}$$

whose vertical maps are pure epimorphisms. Then, we may build the complex F by splicing the short exact sequences in the bottom row of the diagram. Letting P be the chain complex which is built by splicing the split short exact sequences in the top of the diagram (so that $P_n = Q_{n+1} \oplus Q_n$ for all n), the linear maps π_n are the components of a chain map π, as needed.

We fix a chain map $\pi : P \longrightarrow F$ as above and consider a right bounded complex of finitely presented modules C. Then, our assumption implies that any chain map $f : C \longrightarrow F$ can be factored through P as the composition $C \stackrel{g}{\longrightarrow} P \stackrel{\pi}{\longrightarrow} F$, for a suitable chain map g. Since C is a right bounded complex of (finitely presented and hence) finitely generated modules, the following lemma implies that g factors through a right bounded contractible subcomplex $P' \subseteq P$, which consists of finitely presented modules.

Lemma 2.4. *Let C be a complex of finitely generated modules, P a contractible complex all of whose syzygy modules are direct sums of finitely presented modules (so that P is a complex consisting of direct sums of finitely presented modules in each degree) and consider a chain map $g : C \longrightarrow P$. Then, there exists a contractible subcomplex $P' \subseteq P$, which consists of finitely presented modules in each degree, such that $\operatorname{im} g \subseteq P'$. If the complex C is, in addition, left bounded (resp. right bounded, resp. bounded), then the subcomplex P' may be chosen to be left bounded (resp. right bounded, resp. bounded) as well.*

Proof. We may assume that there are modules $(Q_n)_n$, which are all direct sums of finitely presented modules, such that $P_n = Q_n \oplus Q_{n-1}$ and the differential $\partial^P : P_n \longrightarrow P_{n-1}$ is the map $Q_n \oplus Q_{n-1} \longrightarrow Q_{n-1} \oplus Q_{n-2}$, given by $(x_n, x_{n-1}) \mapsto (x_{n-1}, 0)$, $(x_n, x_{n-1}) \in Q_n \oplus Q_{n-1}$ for all n. Since C is a complex of finitely generated modules, we can find finitely presented submodules $Q'_n, Q''_n \subseteq Q_n$, such that $g(C_n) \subseteq Q'_n \oplus Q''_{n-1}$ for all n. (In the case where $C_n = 0$, we may choose $Q'_n = 0 = Q''_{n-1}$.) We now consider for all n a finitely presented submodule $Q'''_n \subseteq Q_n$, such that $Q'_n, Q''_n \subseteq Q'''_n$. (In the case where $Q'_n = Q''_n = 0$, we may choose $Q'''_n = 0$.) We note that $g(C_n) \subseteq Q'''_n \oplus Q'''_{n-1}$ for all n and hence g factors through the contractible subcomplex of P, whose module of n-chains is the finitely presented submodule $Q'''_n \oplus Q'''_{n-1} \subseteq Q_n \oplus Q_{n-1} = P_n$ for all n. □

Proof of Proposition 2.2. (cont.) (v)→(vi): Let us consider a chain map $C \longrightarrow F$, where C is a bounded complex of finitely presented modules. In view of (v), we may factor that chain map through a right bounded contractible complex of finitely presented modules P, as the composition $C \xrightarrow{g} P \longrightarrow F$, for a suitable chain map g. Since C is actually bounded, Lemma 2.4 implies that the subcomplex $\operatorname{im} g$ is contained in a bounded contractible subcomplex of P, which consists of finitely presented modules as well.

(vi)→(vii): Following the technique used by Govorov [9] for flat modules and Neeman [14] for complexes of flat modules, we consider the small category $\mathfrak{C} = \mathfrak{C}_F$, whose objects are the pairs of the form (P, f), where P is a bounded contractible complex of finitely presented modules and $f : P \longrightarrow F$ is a chain map. In order to ensure that the category $\mathfrak{C}$ is small, we also demand that the chain modules of the complex P are all quotients of the countable direct sum $R^{(\mathbb{N})}$ of copies of the regular module. The morphisms in $\mathfrak{C}$ from (P, f) to (P', f') are those chain maps $g : P \longrightarrow P'$, which are such that $f = f'g$. We also consider the functor $\Gamma = \Gamma_F$ from $\mathfrak{C}$ to the category of chain complexes, which maps any object (P, f) onto the chain complex P and any morphism $g : (P, f) \longrightarrow (P', f')$ of $\mathfrak{C}$ onto g (viewed simply as a chain map $g : P \longrightarrow P'$). Let

$$\zeta = \zeta_F : \varinjlim \Gamma \longrightarrow F$$

be the chain map, which is such that the composition

$$P = \Gamma(P, f) \xrightarrow{\eta_{(P,f)}} \varinjlim \Gamma \xrightarrow{\zeta} F$$

coincides with f for any object (P, f) of $\mathfrak{C}$; here, for any object (P, f) we denote by $\eta_{(P,f)}$ the canonical map to the colimit.

Lemma 2.5. *Let F be any complex and consider the category $\mathfrak{C}$ constructed above. Then:*
(i) the category $\mathfrak{C}$ satisfies condition (filt-1) in the definition of a filtered category and
(ii) the chain map ζ defined above is an isomorphism.

Proof. (i) Given two objects (P, f) and (P', f') of $\mathfrak{C}$, we may consider the chain map $[f, f'] : P \oplus P' \longrightarrow F$, which is given in each degree n by the linear map $[f_n, f'_n] : P_n \oplus P'_n \longrightarrow F_n$. Then, the pair $(P \oplus P', [f, f'])$ is an object of $\mathfrak{C}$ and the canonical maps $\iota : P \longrightarrow P \oplus P'$ and

$\iota' : P' \longrightarrow P \oplus P'$ are morphisms in $\mathfrak{C}$, in view of the commutative diagram

$$\begin{array}{ccccc} P & \stackrel{\iota}{\longrightarrow} & P \oplus P' & \stackrel{\iota'}{\longleftarrow} & P' \\ f \downarrow & & \downarrow [f,f'] & & \downarrow f' \\ F & = & F & = & F \end{array}$$

It follows readily that $\mathfrak{C}$ satisfies condition (filt-1).

(ii) First of all, we shall prove that ζ is surjective in each degree. We consider an integer n and a chain $x_n \in F_n$. We also consider the contractible chain complex P, which consists of R in degrees n and $n-1$ and 0's elsewhere, with differential in degree n given by the identity map of R. Then, there exists a unique chain map $f : P \longrightarrow F$, which maps the chain $1 \in P_n$ onto $x_n \in F_n$. As the composition

$$P \stackrel{\eta_{(P,f)}}{\longrightarrow} \varinjlim \Gamma \stackrel{\zeta}{\longrightarrow} F$$

is the chain map f and $x_n \in \operatorname{im} f$, it follows that $x_n \in \operatorname{im} \zeta$, as needed.

We shall now prove that ζ is injective in each degree. To that end, we consider an integer n and an n-chain γ_n of the complex $\varinjlim \Gamma$, which is contained in $\ker \zeta$. Since $\mathfrak{C}$ satisfies condition (filt-1), there exists an object (Q, g) of $\mathfrak{C}$ and a chain $t_n \in Q_n$, such that $\gamma_n = \eta_{(Q,g)}(t_n)$. As the composition

$$Q \stackrel{\eta_{(Q,g)}}{\longrightarrow} \varinjlim \Gamma \stackrel{\zeta}{\longrightarrow} F$$

is the chain map g, we conclude that $g_n(t_n) = 0 \in F_n$. As above, we consider the contractible chain complex P, which consists of R in degrees n and $n-1$ (and 0's elsewhere) with differential in degree n given by the identity map of R, and the unique chain map $h : P \longrightarrow Q$, which maps the chain $1 \in P_n$ onto $t_n \in Q_n$. Then, the commutative diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & P & \stackrel{h}{\longrightarrow} & Q \\ 0 \downarrow & & \downarrow 0 & & \downarrow g \\ F & = & F & = & F \end{array}$$

is a diagram of morphisms in the category $\mathfrak{C}$. Hence, there is an induced commutative diagram of chain complexes (which is part of the colimiting cone defining the complex $\varinjlim \Gamma$)

$$\begin{array}{ccccc} 0 & \longleftarrow & P & \stackrel{h}{\longrightarrow} & Q \\ \| & & \| & & \| \\ \Gamma(0,0) & \longleftarrow & \Gamma(P,0) & \stackrel{\Gamma(h)}{\longrightarrow} & \Gamma(Q,g) \\ \eta_{(0,0)} \downarrow & & \eta_{(P,0)} \downarrow & & \eta_{(Q,g)} \downarrow \\ \varinjlim \Gamma & = & \varinjlim \Gamma & = & \varinjlim \Gamma \end{array}$$

It follows readily that $t_n \in \ker \eta_{(Q,g)}$ and hence $\gamma_n = \eta_{(Q,g)}(t_n) = 0$. □

Proof of Proposition 2.2. (cont.) Invoking Lemma 2.5, it suffices to prove (using our hypothesis that the complex F satisfies condition (vi)) that the category $\mathfrak{C}$ satisfies condition (filt-2) in the definition of a filtered category. To that end, let us consider two objects (P, f) and (P', f') of $\mathfrak{C}$ and two parallel morphisms $a, b : (P, f) \longrightarrow (P', f')$. Then, $a, b : P \longrightarrow P'$ are two chain maps, such that $f'a = f = f'b$. It follows that $f'(b - a) = 0$ and hence f' factors through the cokernel $C = \operatorname{coker}(b - a)$ as the composition $P' \stackrel{p}{\longrightarrow} C \stackrel{\overline{f'}}{\longrightarrow} F$, where p is the quotient map. Since C a bounded complex of finitely presented modules, our hypothesis implies that there exists a bounded contractible complex of finitely presented modules P'', such that $\overline{f'}$ factors as the composition $P' \xrightarrow{\phi} P'' \xrightarrow{f''} F$, for suitable chain maps ϕ and f''. This is all pictured in the commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{a,b} & P' & \xrightarrow{p} & C & \xrightarrow{\phi} & P'' \\ f\downarrow & & f'\downarrow & & \overline{f'}\downarrow & & f''\downarrow \\ F & = & F & = & F & = & F \end{array}$$

Then, the pair (P'', f'') is an object of $\mathfrak{C}$ and the chain map $c = \phi p$ is a morphism in $\mathfrak{C}$ from (P', f') to (P'', f''). Since $c(b-a) = \phi p(b-a) = \phi 0 = 0$, it follows that $cb = ca$.

(vii)→(viii): This is obvious.

(viii)→(i): Assume that F is the filtered colimit of contractible complexes $(P^i)_i$. Since the P^i's are acyclic and homology commutes with filtered colimits, the complex F is acyclic as well. Moreover, for any integer n the short exact sequence

$$0 \longrightarrow Z_nF \longrightarrow F_n \longrightarrow Z_{n-1}F \longrightarrow 0 \tag{6}$$

is the filtered colimit of the split short exact sequences

$$0 \longrightarrow Z_nP^i \longrightarrow P^i_n \longrightarrow Z_{n-1}P^i \longrightarrow 0.$$

It follows readily that the short exact sequence (6) is pure for all n, as needed.

We now consider the special case where F is a complex of flat modules. Since any surjective map onto a flat module is a pure epimorphism, condition (flat-iv) is equivalent to (iv). Hence, it suffices to prove that (flat-iv)→(flat-v)→(flat-vi)→(flat-vii)→(flat-viii)→(viii). The proof of the implication (flat-iv)→(flat-v) is analogous to that of the implication (iv)→(v) given above, by using a surjective chain map from a contractible complex with free syzygy modules onto F and that version of Lemma 2.4, where free (resp. finitely generated free) modules replace the direct sums of finitely presented (resp. the finitely presented) modules. Now, the implications (flat-v)→(flat-vi) and (flat-vi)→(flat-vii) can be proved in the same way as the implications (v)→(vi) and (vi)→(vii) were proved above. Finally, the implications (flat-vii)→(flat-viii) and (flat-viii)→(viii) are obvious. □

The asymmetry in conditions 2.2(iii), 2.2(iv), 2.2(v), 2.2(flat-iv) and 2.2(flat-v), where we only used right bounded complexes, may be remedied, since these conditions turn out to be also equivalent to their left bounded versions. This fact is based on the following general result.

Lemma 2.6. *The following conditions are equivalent for a complex F:*

(i) Any chain map from a right bounded complex of finitely presented modules to F is null-homotopic.

(ii) For any complex of finitely presented modules C, for any chain map $f : C \longrightarrow F$ and for any integer n, there are linear maps $a : C_n \longrightarrow F_{n+1}$ and $b : C_{n-1} \longrightarrow F_n$, such that $f_n = \partial^F a + b\partial^C : C_n \longrightarrow F_n$.[1]

(iii) Any chain map from a left bounded complex of finitely presented modules to F is null-homotopic.

(iv) Any chain map from a bounded complex of finitely presented modules to F is null-homotopic.

Proof. (i)→(ii): Let C be a complex of finitely presented modules and $f : C \longrightarrow F$ a chain map. We fix an integer n and consider the cokernels M, N of the differentials $\partial^C : C_n \longrightarrow C_{n-1}$ and $\partial^F : F_n \longrightarrow F_{n-1}$ respectively. Then, f_{n-1} induces by passage to the quotients a linear map $\phi : M \longrightarrow N$. We also consider the linear map $\theta : N \longrightarrow F_{n-2}$, which is induced by the

[1]The pair (a, b) may be thought of as a "localized in degree n homotopy" between f and the zero map.

differential $\partial^F : F_{n-1} \longrightarrow F_{n-2}$, and note that the composition $N \xrightarrow{\theta} F_{n-2} \xrightarrow{\partial^F} F_{n-3}$ is the zero map. Hence, letting $p : C_{n-1} \longrightarrow M$ be the quotient map, we obtain a chain map

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \xrightarrow{p} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & {\scriptstyle f_{n+1}} \downarrow & & {\scriptstyle f_n} \downarrow & & {\scriptstyle f_{n-1}} \downarrow & & {\scriptstyle \theta\phi} \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & F_{n-2} & \longrightarrow & F_{n-3} & \longrightarrow & \cdots \end{array}$$

Since M is a finitely presented module, we may invoke our hypothesis and conclude that there exist linear maps a and b, as required.

(ii)→(iii): Let C be a left bounded complex of finitely presented modules and consider a chain map $f : C \longrightarrow F$. We shall construct the linear maps $\Sigma_n : C_n \longrightarrow F_{n+1}$, such that $f_n = \partial^F \Sigma_n + \Sigma_{n-1}\partial^C$ for all n, using descending induction on n. Of course, we let $\Sigma_n = 0$ for all $n \gg 0$. Assume that n is an integer and we have already constructed the linear maps $\ldots, \Sigma_{n+2}, \Sigma_{n+1}, \Sigma_n$, in such a way that $f_i = \partial^F \Sigma_i + \Sigma_{i-1}\partial^C$ for all $i > n$. Let L be the cokernel of the differential $\partial^C : C_{n+1} \longrightarrow C_n$ and consider the quotient map $p : C_n \longrightarrow L$ and the linear map $\delta : L \longrightarrow C_{n-1}$, which is induced by $\partial^C : C_n \longrightarrow C_{n-1}$ (so that $\delta p = \partial^C$). Since

$$\begin{array}{rcl} (f_n - \partial^F \Sigma_n)\partial^C & = & f_n \partial^C - \partial^F \Sigma_n \partial^C \\ & = & f_n \partial^C - \partial^F (f_{n+1} - \partial^F \Sigma_{n+1}) \\ & = & f_n \partial^C - \partial^F f_{n+1} + \partial^F \partial^F \Sigma_{n+1} \\ & = & f_n \partial^C - \partial^F f_{n+1} \\ & = & 0, \end{array}$$

there exists a unique linear map $\psi : L \longrightarrow F_n$, which is such that $\psi p = f_n - \partial^F \Sigma_n$. Since

$$\begin{array}{rcl} (\partial^F \psi - f_{n-1}\delta)p & = & \partial^F \psi p - f_{n-1}\delta p \\ & = & \partial^F (f_n - \partial^F \Sigma_n) - f_{n-1}\partial^C \\ & = & \partial^F f_n - \partial^F \partial^F \Sigma_n - f_{n-1}\partial^C \\ & = & \partial^F f_n - f_{n-1}\partial^C \\ & = & 0, \end{array}$$

it follows that $\partial^F \psi - f_{n-1}\delta = 0$ and hence we may consider the chain map

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L & \xrightarrow{\delta} & C_{n-1} & \longrightarrow & C_{n-2} & \longrightarrow & \cdots \\ & & \downarrow & & {\scriptstyle \psi} \downarrow & & {\scriptstyle f_{n-1}} \downarrow & & {\scriptstyle f_{n-2}} \downarrow & & \\ \cdots & \longrightarrow & F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & F_{n-2} & \longrightarrow & \cdots \end{array}$$

Since the module L is finitely presented, we may apply our hypothesis and conclude that there exist linear maps $S : L \longrightarrow F_{n+1}$ and $\Sigma_{n-1} : C_{n-1} \longrightarrow F_n$, such that $\psi = \partial^F S + \Sigma_{n-1}\delta$. Then, the linear maps $\ldots, \Sigma_{n+2}, \Sigma_{n+1}, \Sigma_n + Sp, \Sigma_{n-1}$ complete the inductive step of the construction, since

$$\begin{array}{rcl} \partial^F \Sigma_{n+1} + (\Sigma_n + Sp)\partial^C & = & \partial^F \Sigma_{n+1} + \Sigma_n \partial^C + Sp\partial^C \\ & = & \partial^F \Sigma_{n+1} + \Sigma_n \partial^C + S0 \\ & = & \partial^F \Sigma_{n+1} + \Sigma_n \partial^C \\ & = & f_{n+1} \end{array}$$

and

$$\begin{array}{rcl} \partial^F (\Sigma_n + Sp) + \Sigma_{n-1}\partial^C & = & \partial^F \Sigma_n + \partial^F Sp + \Sigma_{n-1}\partial^C \\ & = & \partial^F \Sigma_n + \partial^F Sp + \Sigma_{n-1}\delta p \\ & = & \partial^F \Sigma_n + (\partial^F S + \Sigma_{n-1}\delta)p \\ & = & \partial^F \Sigma_n + \psi p \\ & = & f_n. \end{array}$$

Note that this construction defines the component of the chain homotopy between f and the zero map at any given degree in two steps: In the first step, one defines an "approximation" (in the argument above, this is the linear map Σ_n coming from the inductive hypothesis). In the second step, one adds a "correction" term (in the argument above, this is the summand Sp), so that the previously established homotopy relations remain valid and the additional homotopy relation in the next degree downwards is introduced.

(iii)$\rightarrow$(iv): This is obvious.

(iv)$\rightarrow$(i): Let C be a right bounded complex of finitely presented modules and consider the subcomplexes $C^{\leq n} \subseteq C$ for all $n \geq 0$. It is easily seen that the inclusions $C^{\leq n} \subseteq C^{\leq n+1}$ induce surjective chain maps between the Hom-complexes

$$\mathrm{Hom}_R\big(C^{\leq n+1}, F\big) \longrightarrow \mathrm{Hom}_R\big(C^{\leq n}, F\big)$$

for all n. Since the complex C is identified with the union (colimit) of the $C^{\leq n}$'s, the limit of the inverse system $\big(\mathrm{Hom}_R\big(C^{\leq n}, F\big)\big)_n$ is identified with the Hom-complex $\mathrm{Hom}_R(C, F)$. We may now use standard facts about inverse systems of complexes with surjective structure maps (cf. [19, §3.5]), in order to obtain a short exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_R(C, F) \longrightarrow \prod\nolimits_n \mathrm{Hom}_R\big(C^{\leq n}, F\big) \longrightarrow \prod\nolimits_n \mathrm{Hom}_R\big(C^{\leq n}, F\big) \longrightarrow 0.$$

In view of our hypothesis, all chain maps from any translate of the bounded complex $C^{\leq n}$ to F are null-homotopic; hence, the complex $\mathrm{Hom}_R\big(C^{\leq n}, F\big)$ is acyclic for all $n \geq 0$. It follows readily that the complex $\mathrm{Hom}_R(C, F)$ is acyclic as well. In particular, any chain map from C to F is null-homotopic, as needed. $\square$

Remark 2.7. The only property of the class of finitely presented modules that was used in the proof of the previous lemma is its closure under cokernels. Hence, for any cokernel-closed class $\mathfrak{A}$ of modules, one has an analogous result concerning complexes of $\mathfrak{A}$-modules which are (left, right or on both sides) bounded.

We can now formulate the left bounded versions of the characterizations of pure acyclic complexes (of flat modules), that were stated as conditions 2.2(iii), 2.2(iv), 2.2(v), 2.2(flat-iv) and 2.2(flat-v).

Proposition 2.8. *The following conditions are equivalent for a complex F:*

(i) F is pure acyclic,

(ii) any chain map from a left bounded complex of finitely presented modules to F is null-homotopic,

(iii) F is acyclic and for any chain map $f : Y \longrightarrow F$, which consists of pure epimorphisms in each degree, and for any left bounded complex of finitely presented modules C, any chain map $C \longrightarrow F$ can be factored through f,

(iv) any chain map from a left bounded complex of finitely presented modules to F can be factored through a left bounded contractible complex of finitely presented modules,

(v) any chain map from a bounded complex of finitely presented modules to F is null-homotopic.

If F is a complex of flat modules, then these conditions are also equivalent to:

(flat-iii) F is acyclic and for any surjective chain map $f : Y \longrightarrow F$ and for any left bounded complex of finitely presented modules C, any chain map $C \longrightarrow F$ can be factored through f,

(flat-iv) any chain map from a left bounded complex of finitely presented modules to F can be factored through a left bounded contractible complex of finitely generated free modules.

Proof. (i)↔(ii)↔(v): Since condition (i) is equivalent to condition 2.2(iii), the equivalence between conditions (i), (ii) and (v) is an immediate consequence of Lemma 2.6.

The proof of the implications (ii)→(iii)→(iv) is exactly the same as the proof of the implications 2.2(iii)→2.2(iv)→2.2(v) in Proposition 2.2.

(iv)→(i): In view of the equivalence between condition (i) and condition 2.2(vi), it suffices to prove that any chain map from a bounded complex of finitely presented modules to F can be factored through a bounded contractible complex of finitely presented modules. Using (iv), the proof of the latter claim is exactly the same as the proof of the implication 2.2(v)→2.2(vi) in Proposition 2.2.

We now consider the special case where F is a complex of flat modules. Since any surjective map onto a flat module is a pure epimorphism, condition (flat-iii) is equivalent to (iii). On the other hand, condition (flat-iv) is a strengthening of (iv) and hence it only suffices to prove that (flat-iii)→(flat-iv). This may be achieved by using the same arguments as in the proof of the implication 2.2(flat-iv)→2.2(flat-v) in Proposition 2.2. □

3. The relation to pure projective modules

In this final section, we relate the pure acyclic complexes to the complexes of pure projective modules. In the case of complexes of flat modules, this relation was established by Neeman [14]; his proof uses the theory of well-generated triangulated categories and Brown representability. The proof presented here avoids these techniques and has a purely algebraic flavor.

Lemma 3.1. *Any complex of pure projective modules is homotopy equivalent to a complex whose chain modules are direct sums of finitely presented modules.*

Proof. Let P be a complex of pure projective modules. Then, using a variation of Eilenberg's trick, we can find for all n a direct sum of finitely presented modules Q_n, which is such that the module $P_n \oplus Q_n$ is a direct sum of finitely presented modules as well. Let $Q(n)$ be the complex which consists of Q_n in degrees n and $n-1$ and 0's elsewhere (with the differential in degree n, given by the identity map of Q_n); then, $Q(n)$ is contractible. If $Q = \bigoplus_n Q(n)$, then Q is a contractible complex as well and the direct sum $P \oplus Q$ consists in any degree n of the module $P_n \oplus Q_n \oplus Q_{n+1}$, which is a direct sum of finitely presented modules. □

Lemma 3.2. *Let P be a non-zero complex, whose chain modules are direct sums of finitely presented modules. For all n we fix a family I_n of finitely presented modules, which is such that $P_n = \bigoplus_{C \in I_n} C$. Then, there exists a non-zero left bounded subcomplex $P' \subseteq P$, which is such that the module of degree n chains P'_n is a finitely presented module of the form $\bigoplus_{C \in I'_n} C$, for a suitable finite subfamily $I'_n \subseteq I_n$ for all n.*

Proof. Since P is a non-zero complex, there exists an integer n_0, such that $P_{n_0} \neq 0$. Then, the family I_{n_0} contains non-zero modules and hence we may choose a submodule $C \in I_{n_0}$ of P_{n_0}, with $C \neq 0$. We let $P'_n = 0$ (and $I'_n = \emptyset$) if $n > n_0$ and define $P'_{n_0} = C$ (and $I'_{n_0} = \{C\}$). We now use descending induction and construct finitely presented submodules $P'_n \subseteq P_n$, such that $P'_n = \bigoplus_{C \in I'_n} C$ for a suitable finite subfamily $I'_n \subseteq I_n$ and $\partial^P(P'_n) \subseteq P'_{n-1}$ for all $n \leq n_0$. The P'_n's define a subcomplex $P' \subseteq P$ that satisfies the requirements in the statement. □

We shall also need the following couple of lemmas, which are essentially due to Neeman [14]. We present his elegant arguments below (in a slightly more general context).

Lemma 3.3. *Let M, N, F be three modules and assume that M (resp. N, resp. F) is finitely generated (resp. finitely presented, resp. flat). We consider two linear maps $f : M \longrightarrow N$ and $g : N \longrightarrow F$ and assume that $gf = 0$. Then, there exists a finitely generated free module P*

and a factorization of g as the composition $N \xrightarrow{g'} P \xrightarrow{g''} F$, for suitable linear maps g' and g'', where g' is such that $g'f = 0$.

Proof. Since $gf = 0$, the linear map g may be factored through the cokernel $\overline{N} = \operatorname{coker} f$, as the composition $N \xrightarrow{p} \overline{N} \xrightarrow{\overline{g}} F$, where p is the quotient map. The module $\overline{N}$ being finitely presented, [12, Lemme 1.1] implies the existence of a finitely generated free module P, such that $\overline{g}$ factors as the composition $\overline{N} \xrightarrow{a} P \xrightarrow{b} F$, for suitable linear maps a and b. Then, g factors as the composition $N \xrightarrow{ap} P \xrightarrow{b} F$ and we have $(ap)f = a(pf) = a0 = 0$. □

Lemma 3.4. *Let C be a left bounded complex of finitely presented modules and consider a complex F of flat modules and a chain map $f : C \longrightarrow F$. Then, there exists a left bounded complex of finitely generated free modules P and a factorization of f as the composition*

$$C \xrightarrow{a} P \xrightarrow{b} F,$$

for suitable chain maps a and b.

Proof. Since $C_n = 0$ for all $n \gg 0$, we may set $P_n = 0$ for all $n \gg 0$. We shall define a complex P and a factorization of f as in the statement, using descending induction on n. We thus assume that n is an integer, such that the required factorization has been constructed in degrees $i \geq n$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial^C} & C_{n+1} & \xrightarrow{\partial^C} & C_n & \xrightarrow{\partial^C} & C_{n-1} & \xrightarrow{\partial^C} & \cdots \\
 & & \downarrow a_{n+1} & & \downarrow a_n & & & & \\
\cdots & \xrightarrow{\partial^P} & P_{n+1} & \xrightarrow{\partial^P} & P_n & & & & \\
 & & \downarrow b_{n+1} & & \downarrow b_n & & & & \\
\cdots & \xrightarrow{\partial^F} & F_{n+1} & \xrightarrow{\partial^F} & F_n & \xrightarrow{\partial^F} & F_{n-1} & \xrightarrow{\partial^F} & \cdots
\end{array}$$

We consider the cokernel $\overline{P}_n$ of the linear map $\partial^P : P_{n+1} \longrightarrow P_n$ and let $p : P_n \longrightarrow \overline{P}_n$ be the quotient map. Since $\partial^F b_n \partial^P = \partial^F \partial^F b_{n+1} = 0 b_{n+1} = 0$, there exists a unique linear map $\gamma : \overline{P}_n \longrightarrow F_{n-1}$, such that $\gamma p = \partial^F b_n$. Then, $\gamma p a_n = \partial^F b_n a_n = \partial^F f_n = f_{n-1}\partial^C$ and hence the composition

$$C_n \xrightarrow{(pa_n, -\partial^C)} \overline{P}_n \oplus C_{n-1} \xrightarrow{[\gamma, f_{n-1}]} F_{n-1}$$

is the zero map. Since C_n and $\overline{P}_n \oplus C_{n-1}$ are finitely presented modules, Lemma 3.3 implies that there exists a finitely generated free module P_{n-1} and a factorization of $[\gamma, f_{n-1}]$ as the composition

$$\overline{P}_n \oplus C_{n-1} \xrightarrow{[\lambda, a_{n-1}]} P_{n-1} \xrightarrow{b_{n-1}} F_{n-1},$$

for suitable linear maps λ, a_{n-1} and b_{n-1}, which are such that the composition

$$C_n \xrightarrow{(pa_n, -\partial^C)} \overline{P}_n \oplus C_{n-1} \xrightarrow{[\lambda, a_{n-1}]} P_{n-1}$$

is the zero map. Then, $b_{n-1}\lambda = \gamma$ (and hence $b_{n-1}\lambda p = \gamma p = \partial^F b_n$), $b_{n-1}a_{n-1} = f_{n-1}$ and $\lambda p a_n = a_{n-1}\partial^C$. As we also have $\lambda p \partial^P = \lambda 0 = 0$, the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial^C} & C_{n+1} & \xrightarrow{\partial^C} & C_n & \xrightarrow{\partial^C} & C_{n-1} & \xrightarrow{\partial^C} & \cdots \\
 & & \downarrow a_{n+1} & & \downarrow a_n & & \downarrow a_{n-1} & & \\
\cdots & \xrightarrow{\partial^P} & P_{n+1} & \xrightarrow{\partial^P} & P_n & \xrightarrow{\lambda p} & P_{n-1} & & \\
 & & \downarrow b_{n+1} & & \downarrow b_n & & \downarrow b_{n-1} & & \\
\cdots & \xrightarrow{\partial^F} & F_{n+1} & \xrightarrow{\partial^F} & F_n & \xrightarrow{\partial^F} & F_{n-1} & \xrightarrow{\partial^F} & \cdots
\end{array}$$

completes the inductive step of the construction. □

Remark 3.5. We adopt the notation of Lemma 3.4 and assume that the complex C therein is bounded. It is worth pointing out that the complex P constructed in the proof above is left bounded, but not necessarily bounded (even if C is a complex concentrated in one degree).

Theorem 3.6. *The following conditions are equivalent for a complex F:*
(i) F is pure acyclic,
(ii) any chain map from a complex of pure projective modules to F is null-homotopic.
If F is a complex of flat modules, then these conditions are also equivalent to:
(flat-ii) any chain map from a complex of projective modules to F is null-homotopic,
(flat-iii) any chain map from a left bounded complex of finitely generated projective modules to F is null-homotopic.

Proof. (i)→(ii): Let P be a complex of pure projective modules and $f : P \longrightarrow F$ a chain map. We have to prove that f is null-homotopic. In view of Lemma 3.1, we may assume that P_n is the direct sum of a family I_n of finitely presented modules (i.e. that $P_n = \bigoplus_{C \in I_n} C$) for all n. Let $\mathfrak{X}$ be the set consisting of the triples (P', I', Σ'), where:
(a) P' is a subcomplex of P, whose module of degree n chains is of the form $P'_n = \bigoplus_{C \in I'_n} C$, for a suitable subfamily $I'_n \subseteq I_n$ for all n,
(b) I' is the disjoint union of the families $(I'_n)_n$ that were considered in (a) above and
(c) $\Sigma' : f|_{P'} \sim 0$ is a homotopy from the restriction $f|_{P'}$ of f to the zero map.
We may order the set $\mathfrak{X}$, in such a way that for any $(P', I', \Sigma'), (P'', I'', \Sigma'') \in \mathfrak{X}$ we have

$$(P', I', \Sigma') \leq (P'', I'', \Sigma'') \iff P' \subseteq P'',\ I' \subseteq I'' \ \text{ and } \ \Sigma' = \Sigma''|_{P'}.$$

The ordered set $\mathfrak{X}$ is non-empty, as $(0, \emptyset, 0) \in \mathfrak{X}$, and any linearly ordered subset therein has an upper bound (the union). Therefore, we may apply Zorn's lemma and conclude that there exists a maximal element $(P', I', \Sigma') \in \mathfrak{X}$. We shall prove that $P' = P$; this will imply that Σ' is the required homotopy from $f|_{P'} = f|_P = f$ to the zero map.

Assume on the contrary that P' is a proper subcomplex of P. Then, the quotient complex P/P' is non-zero and its chain modules are direct sums of finitely presented modules; in fact, we have $P_n/P'_n = \bigoplus_{C \in I_n \setminus I'_n} C$ for all n. Invoking Lemma 3.2, we conclude that there exists a subcomplex $P'' \subseteq P$ with $P'' \supseteq P'$, such that the quotient complex P''/P' is a non-zero left bounded subcomplex of P/P', whose module of degree n chains is a finitely presented module of the form $P''_n/P'_n = \bigoplus_{C \in J_n} C$, for a suitable finite subfamily $J_n \subseteq I_n \setminus I'_n$ for all n. Then, the chain modules of the complex P'' are direct sums of finitely presented modules; in fact, we have $P''_n = \bigoplus_{C \in I''_n} C$, where $I''_n = I'_n \cup J_n$ for all n. We now consider the short exact sequence of complexes

$$0 \longrightarrow P' \xrightarrow{\iota} P'' \xrightarrow{p} P''/P' \longrightarrow 0,$$

where ι (resp. p) denotes the inclusion (resp. the quotient map). The corresponding short exact sequences of chain modules are all split and hence there is an induced short exact sequence of complexes of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(P''/P', F) \xrightarrow{p^*} \mathrm{Hom}_R(P'', F) \xrightarrow{\iota^*} \mathrm{Hom}_R(P', F) \longrightarrow 0.$$

The homotopy Σ' from the restriction $f|_{P'}$ to the zero map is a certain homogeneous linear map $P' \longrightarrow F$ of degree 1. Since the map ι^* is onto, it follows that there exists a homogeneous linear map $\Sigma'' : P'' \longrightarrow F$ of degree 1, which is such that $\Sigma''\iota = \Sigma'$ (i.e. for which $\Sigma''|_{P'} = \Sigma'$). Then, the chain map $g = f|_{P''} - (\partial^F \Sigma'' + \Sigma'' \partial^{P''}) : P'' \longrightarrow F$ vanishes on P' (since $f|_{P'} = \partial^F \Sigma' + \Sigma' \partial^{P'}$) and hence defines by passage to the quotient a chain map $h : P''/P' \longrightarrow F$ (so that we have an equality $g = hp$). Since we already know that condition (i) is equivalent to condition 2.8(ii), we may conclude that the chain map h is null-homotopic; thus, there exists a homogeneous linear map $S : P''/P' \longrightarrow F$ of degree 1, such that $h = \partial^F S + S\overline{\partial}$ (where $\overline{\partial}$ denotes the differential of the quotient complex P''/P'). We now consider the homogeneous linear map $\Sigma'' + Sp : P'' \longrightarrow F$ of degree 1 and compute

$$\begin{aligned} \partial^F(\Sigma'' + Sp) + (\Sigma'' + Sp)\partial^{P''} &= \partial^F\Sigma'' + \Sigma''\partial^{P''} + \partial^F Sp + Sp\partial^{P''} \\ &= \partial^F\Sigma'' + \Sigma''\partial^{P''} + \partial^F Sp + S\overline{\partial}p \\ &= \partial^F\Sigma'' + \Sigma''\partial^{P''} + (\partial^F S + S\overline{\partial})p \\ &= \partial^F\Sigma'' + \Sigma''\partial^{P''} + hp \\ &= \partial^F\Sigma'' + \Sigma''\partial^{P''} + g \\ &= f|_{P''} . \end{aligned}$$

Therefore, letting I'' be the disjoint union of the I''_n's, we obtain an element $(P'', I'', \Sigma'' + Sp)$ of $\mathfrak{X}$. The restriction of the homotopy $\Sigma'' + Sp$ to the subcomplex $P' \subseteq P''$ is the homotopy

$$(\Sigma'' + Sp)\iota = \Sigma''\iota + Sp\iota = \Sigma''\iota + S0 = \Sigma''\iota = \Sigma'$$

and hence $(P', I', \Sigma') \leq (P'', I'', \Sigma'' + Sp)$. Since the quotient complex P''/P' is non-zero, the complex P'' contains properly P'. It follows that the element $(P'', I'', \Sigma'' + Sp)$ is strictly bigger than the maximal element (P', I', Σ') in the ordered set $\mathfrak{X}$, which is absurd. Thus, the assumption that P' is a proper subcomplex of P leads to a contradiction; we conclude that $P' = P$, as needed.

(ii)$\rightarrow$(i): This is obvious, since condition (ii) is a strengthening of condition 2.8(v), which is itself equivalent to (i), as shown in Proposition 2.8.

We now consider the special case where F is a complex of flat modules. As the implications (ii)$\rightarrow$(flat-ii)$\rightarrow$(flat-iii) are obvious, it only suffices to prove that (flat-iii)$\rightarrow$(i). To that end, we shall prove that condition (flat-iii) implies condition 2.8(ii), i.e. that any chain map from a left bounded complex of finitely presented modules to F is null-homotopic. In view of Proposition 2.8, this will complete the proof. We therefore fix a left bounded complex of finitely presented modules C and consider a chain map $f : C \longrightarrow F$. Then, Lemma 3.4 implies that there exists a left bounded complex of finitely generated projective modules P and a factorization of f as the composition $C \xrightarrow{a} P \xrightarrow{b} F$, for suitable chain maps a and b. As our assumption implies that b is null-homotopic, it follows that $f = ba$ is null-homotopic as well. □

Corollary 3.7. *A complex of pure projective modules is pure acyclic if and only if it is contractible.*

Proof. Let F be a pure acyclic complex of pure projective modules. In view of Theorem 3.6, the identity map $1_F : F \longrightarrow F$ is then null-homotopic and hence F is contractible.

Conversely, if F is a contractible complex, then the complex DF is contractible as well. It follows that condition 1.1(ii) is satisfied and hence F is pure acyclic. □

Corollary 3.8. *(Simson [18]) Let M be a module and assume that there exists a positive integer n and a pure acyclic complex*

$$0 \longrightarrow M \xrightarrow{\eta} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{p} M \longrightarrow 0,$$

where the modules P_i are pure projective for all $i = 0, 1, \ldots, n-1$. Then, M is pure projective as well.

Proof. We may splice copies of the given exact sequence and obtain a (doubly infinite) pure acyclic complex of pure projective modules

$$\cdots \longrightarrow P_0 \xrightarrow{\eta p} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\eta p} P_{n-1} \longrightarrow \cdots .$$

Invoking Corollary 3.7, it follows that the above complex is contractible and hence the module $M = \operatorname{im} \eta p$ is a direct summand of P_0. In particular, M is pure projective. □

Remark 3.9. Corollary 3.7 implies that the pure acyclic complexes of projective modules are necessarily contractible; as noted by Neeman in [14, Remark 2.15], this follows from the part of Theorem 3.6 that concerns pure acyclic complexes of flat modules. Then, the analogue of Corollary 3.8 shows that any periodic flat module is projective, a result which is due to Benson and Goodearl [3]; this was noted by Christensen and Holm in [6, Proposition 7.6].

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