Dyadic A_1 weights and equimeasurable rearrangements of functions

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Abstract: We prove that the non-increasing rearrangement of a dyadic A_1-weight w with dyadic A_1 constant $[w]_1^{\mathcal{T}} = c$ with respect to a tree $\mathcal{T}$ of homogeneity k, on a non-atomic probability space, is a usual A_1 weight on $(0,1]$ with A_1-constant $[w^*]_1$ not more than $kc-k+1$. We prove also that the result is sharp, when one considers all such weights w.

1. Introduction

The theory of Muckenhoupt weights has been proved to be an important tool in analysis due to their self-improving properties (see [2], [3] and [9]). One class of special interest is $A_1(J,c)$ where J is an interval on $\mathbb{R}$ and c is a constant such that $c \geq 1$. Then $A_1(J,c)$ is defined as the class of all non-negative locally integrable functions w defined on J, such that for every subinterval $I \subseteq J$ we have that

$$\frac{1}{|I|}\int_I w(y)dy \leq c \operatorname*{ess\,inf}_{x\in I} w(x) \tag{1.1}$$

where $|\cdot|$ is the Lesbesgue measure on $\mathbb{R}$.

In [1] it is proved that if $w \in A_1(J,c)$ then $w^* \in A_1((0,|J|],c)$, where w^* is the non-increasing rearrangement of w. That is for every $w \in A_1(J,c)$ the following inequality is satisfied

$$\frac{1}{t}\int_0^t w^*(y)dy \leq c\, w^*(t), \tag{1.2}$$

for every $t \in (0,|J|]$. Here for a $w : J \to \mathbb{R}^+$, w^* is defined by the following way. By denoting $A_w(y) = [x \in J : |w(x)| > y]$ and $m_w(y) = |A_w(y)|$ the distribution function

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of $|w|$ then w^* is given by $w^*(t) = inf(y > 0 : m_w(y) < t)$. An equivalent formulation of the non-increasing rearrangement can be given as follows

$$w^*(t) = \sup_{\substack{e \subseteq J \\ |e| \geq t}} \inf_{x \in e} |w(x)|, \quad \text{for any} \quad t \in (0, |J|].$$

It is well known that the function w^* which is defined on $(0, |J|]$, is non-increasing, non negative and equimeasurable to $|w|$. Inequality (1.2) is the tool as one can see in [1], in the determination of all p such that $p > 1$ and $w \in RH_p^J(c')$ for some $1 \leq c' < +\infty$ whenever $w \in A_1(J, c)$. Here by $RH_p^J(c')$ we mean the class of all weights w defined on J which satisfy a reverse Holder inequality with constant c' upon all the subintervals $I \subseteq J$. One can also see related problems for estimates for the range of p in higher dimensions in [4] and [5]. For related results one can see also [6], [10] and [11].

In this paper we are interested for those weights w defined on a dyadic cube Q on R^n or on the whole R^n satisfying condition (1.1) for all dyadic subcubes of it's domain. More precisely, a locally integrable non-negative function w on $\mathbb{R}^n$ is called a dyadic A_1 weight if it satisfies the following condition

$$\frac{1}{|Q|} \int_Q w(y) dy \leq c \operatorname{ess} \inf_{x \in Q} w(x), \tag{1.3}$$

for every dyadic cube Q on $\mathbb{R}^n$.

This condition is equivalent to the inequality

$$\mathcal{M}_d w(x) \leq c\, w(x), \tag{1.4}$$

for almost all $x \in \mathbb{R}^n$. Here $\mathcal{M}_d$ is the dyadic maximal operator defined by

$$\mathcal{M}_d w(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |w(y)| dy : x \in Q,\ Q \subset \mathbb{R}^n \text{ is a dyadic cube} \right\}. \tag{1.5}$$

The smallest $c \geq 1$ for which (1.3) (equivalently (1.4)) holds is called the dyadic A_1 constant of w and is denoted by $[w]_1^d$.

Let us now fix such a weight w. In [7] it is proved that it belongs to L^p for any $p \in [1, p_0(n, c))$ where,

$$p_0(n, c) = \frac{log(2^n)}{log[2^n - (2^n - 1)c^{-1}]}$$

.

Moreover it satisfies a reverse Hölder inequality for all p in the above range upon all dyadic cubes on R^n. More precisely the following is true as can be seen in [7].

Theorem 1. *Let w be a dyadic A_1 weight defined on $\mathbb{R}^n$ with dyadic A_1 constant $[w]_1^d = c$. Then the following inequality is true*

$$\frac{1}{|Q|}\int_Q (\mathcal{M}_d w)^p \leq \frac{2^n - 1}{2^n - [2^n - (2^n - 1)c^{-1}]^p}\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)^p$$

for every Q dyadic cube on R^n and p in the range $[1, p_0(n,c))$. Additionally the above inequality is sharp for any fixed $c \geq 1$ and p in the above range.

Theorem 1 now implies that the range of p's mentioned above is best possible. Let now w be a weight defined on a dyadic cube $Q \subset R^n$ which satisfies the A_1 condition upon all dyadic subcubes of Q with constant not more than $c > 1$. Then as it is mentioned in [7] it's non-increasing rearrangement w^* does not necessarily belong to $A_1((0,|Q|],c)$. As a result certain questions arise: Does w^* belongs to $A_1((0,|Q|],c')$ for some $c' \geq c$ and is there an upper bound on these c' ? What is the least one ? These questions are answered by the following

Theorem 2. *Let w be a dyadic A_1 weight on $\mathbb{R}^n$ with dyadic A_1 constant $[w]_1^d = c$. Let Q be a fixed dyadic cube on $\mathbb{R}^n$. Then if we denote by w/Q the restriction of w on Q, the following inequality is satisfied*

$$\frac{1}{t}\int_0^t (w/Q)^*(y)dy \leq (2^n c - 2^n + 1)(w/Q)^*(t), \tag{1.6}$$

for every $t \in (0,|Q|]$. Moreover the last inequality is sharp when one considers all dyadic A_1 weights with $[w]_1^d = c$.

We remark that by using a standard dilation argument it suffices to prove (1.6) for $Q = [0,1]^n$ and for all functions w defined only on $[0,1]^n$ and satisfying the A_1 condition only for dyadic cubes contained in $[0,1]^n$. Actually, we will work on more general non-atomic probability spaces (X,μ) equipped with a structure $\mathcal{T}$ similar to the dyadic one. (We give the precise definition in the next section).

The paper is organized as follows: In Section 2. we give some tools needed for the proof of Theorem 2. These are obtained from [7] and [8]. In Section 3 we give the proof of Theorem 2 in it's general form (as Theorem 3) and mention two applications of it.

2. Preliminaries

We fix a non-atomic probability space (X,μ) and a positive integer $k \geq 2$. We give the following

Definition 1. *A set of measurable subsets of X will be called a tree of homogeneity k if*

i) For every $I \in \mathcal{T}$ there corresponds a subset $C(I) \subseteq \mathcal{T}$ containing exactly k pairwise disjoint subsets of I such that $I = \cup C(I)$ and each element of $C(I)$ has measure $(1/k)\mu(I)$.

ii) $\mathcal{T} = \bigcup\limits_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup\limits_{I \in \mathcal{T}_{(m)}} C(I)$.

iii) The tree $\mathcal{T}$ differentiates $L^1(X, \mu)$, that is if $\varphi \in L^1(X, \mu)$ then for μ-almost all $x \in X$ and every sequence $(I_k)_{k \in \mathbb{N}}$ such that $x \in I_k$, $I_k \in \mathcal{T}$ and $\mu(I_k) \to 0$ we have that

$$\varphi(x) = \lim_{k \to +\infty} \frac{1}{\mu(I_k)} \int_{I_k} \varphi d\mu.$$

It is clear that each family $\mathcal{T}_{(m)}$ consists of k^m pairwise disjoint sets, each having measure k^{-m}, whose union is X. Moreover, if $I, J \in \mathcal{T}$ and $I \cap J$ is non empty then $I \subseteq J$ or $J \subseteq I$.

For this family $\mathcal{T}$ we define the associated maximal operator $\mathcal{M}_{\mathcal{T}}$ by

$$\mathcal{M}_{\mathcal{T}}\varphi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\varphi| d\mu : x \in I \in \mathcal{T} \right\}, \tag{2.1}$$

and for any $\varphi \in L^1(X, \mu)$ and we will say that a non-negative integrable function w is an A_1 weight with respect to $\mathcal{T}$ if

$$\mathcal{M}_{\mathcal{T}}\varphi(x) \leq C\varphi(x), \tag{2.2}$$

for almost every $x \in X$. The smallest constant C for which (2.2) holds will be called the A_1 constant of w with respect to $\mathcal{T}$ and will be denoted by $[w]_1^{\mathcal{T}}$. We give now the following:

Definition 2. *Every non-constant function w of the form $w = \sum\limits_{P \in \mathcal{T}_{(m)}} \lambda_P x_P$, for a specific $m > 0$, and for positive λ_P, will be called a $\mathcal{T}$-step function (x_P denotes the characteristic function of P).*

It is then clear that every $\mathcal{T}$-step function is an A_1 weight with respect to $\mathcal{T}$. Let now w be a weight as in Definition 2. Let also $[w]_1^{\mathcal{T}} = c > 1$ and for any $I \in \mathcal{T}$ write $Av_I(w) = \frac{1}{\mu(I)} \int\limits_I w d\mu$.

Now for every $x \in X$, let $I_w(x)$ denote the largest element of the set $\{I \in \mathcal{T} : x \in I$ and $\mathcal{M}_{\mathcal{T}}w(x) = Av_I(w)\}$ (which is non-empty since $Av_J(w) = Av_P(w)$ for every $P \in \mathcal{T}_{(m)}$ and $J \subseteq P$).

Next for any $I \in \mathcal{T}$ we define the set

$$A_I = A(w, I) = \{x \in X : I_w(x) = I\}$$

and let $S = S_w$ denote the set of all $I \in \mathcal{T}$ such that A_I is non-empty. It is clear that each such A_I is a union of certain P from $\mathcal{T}_{(m)}$ and moreover

$$\mathcal{M}_{\mathcal{T}} w = \sum_{I \in S} Av_I(w) x_{A_I}.$$

We also define the correspondence $I \to I^*$ with respect to S as follows: I^* is the smallest element of $\{J \in S_w : I \subsetneq J\}$. This is defined for every $I \in S$ that is not maximal with respect to $\subseteq$.

We recall now two Lemmas from [7] and for the sake of completness we present their proof.

Lemma 1. *Let w be as above. Then for all $I \in \mathcal{T}$ we have $I \in S$, if and only if, $Av_Q(w) < Av_I(w)$ whenever $I \subseteq Q \in \mathcal{T}$, $I \neq Q$. In particular $X \in S$ and so $I \to I^*$ is defined for all $I \in S$ such that $I \neq X$.*

Proof. If $I \in S$ then it is clear that the condition that is described above holds. Let now $I \in \mathcal{T}$ for which $Av_Q(w) < Av_I(w)$ for any Q that strictly contains I and belongs to the tree $\mathcal{T}$. Assume that $I \in \mathcal{T}_{(s)}$, then since

$$Av_J(w) = \frac{\sum_{F \in C(J)} \mu(F) Av_F(w)}{\sum_{F \in C(J)} \mu(F)}$$

we conclude that for each $J \in \mathcal{T}$ there exists $F \in C(J)$ such that $Av_F(w) \leq Av_J(w)$. Applying the above $m - s$ times we get a chain $I = I_0 \supset I_1 \supset I_2 \supset ... \supset I_{m-s}$ such that $I_r \in \mathcal{T}_{(s+r)}$ for each r and moreover $Av_{I_{m-s}}(w) \leq Av_{I_{m-s-1}}(w) \leq ... \leq Av_{I_1}(w) \leq Av_{I_0}(w) = Av_I(w)$. Now because on the assumption on I and the last mentioned inequalities we conclude that $I_w(x) = I$ for every $x \in I_{m-s}$, therefore $I \in S$.

In the following denote by y_I the $Av_I(w)$ for any $I \in \mathcal{T}$.

Lemma 2. *Let $I \in S$. Then, if $J \in S$ is such that $J^* = I$, then $y_I < y_J \leq (k - (k - 1)c^{-1}) y_I$.*

Proof. The inequality on the left follows immediately from Lemma 1. Consider now the unique $F \in \mathcal{T}$ such that $J \in C(F)$. Obviously $F \subseteq I$. It is also true that $Av_F(w) \leq y_I$. Indeed $I \in S$ implies that $Av_Q(w) < y_I$ whenever $I \subseteq Q$, $I \neq Q$ and so if $Av_F(w) > y_I$ there would exist $F_1 \in \mathcal{T}$ such that $F \subseteq F_1 \subseteq I$ with $F_1 \neq I$ and $Av_{F_1}(w) > Av_Q(w)$

whenever $F_1 \subseteq Q, F_1 \neq Q$. This combined with Lemma 1 implies that F_1 must lie in S, which doesn't agree with our hypothesis that $J^* = I$. Now note that for every x that belongs to the set-theoretic difference $F \setminus J$ we have $[w]_1^{\mathcal{T}} w(x) \geq \mathcal{M}_{\mathcal{T}} w(x) \geq y_I$, hence integrating over $F \setminus J$ and using all the above we get

$$y_I \geq Av_F(w) \geq \frac{\mu(J)}{\mu(F)} y_J + \frac{\mu(F \setminus J)}{\mu(F)} \frac{1}{[w]_1^{\mathcal{T}}} y_I = \frac{y_J + (k-1)c^{-1} y_I}{k}$$

and from this we immediately conclude the right inequality that is stated in this Lemma.

3. Main theorem and proof

In this section we will prove the following.

Theorem 3. *Let $\mathcal{T}$ be a tree of homogeneity $k \geq 2$ on the probability non-atomic space (X, μ), and let w be A_1 weight with respect to $\mathcal{T}$ with A_1-constant $\left[w\right]_1^{\mathcal{T}} = c$. Then if one considers $w^* : (0,1] \to \mathbb{R}^+$ the non-increasing rearrangement of w we have that $\frac{1}{t} \int\limits_0^t w^*(y)dy \leq (kc - k + 1)w^*(t)$, for every $t \in (0,1]$. Moreover the constant appearing in the right of the last inequality is sharp, if one considers all such weights with A_1-constant with respect to $\mathcal{T}$ equal to c.*

Proof. We suppose for the beginning that w is a $\mathcal{T}$-step function. Fix $t \in (0,1]$ and consider the set

$$\begin{aligned} E_t &= \{x \in X : \mathcal{M}_{\mathcal{T}} w(x) > c\, w^*(t)\} \\ &= \{\mathcal{M}_{\mathcal{T}} w > c\lambda\}, \quad \text{where} \quad \lambda = w^*(t). \end{aligned}$$

Then E_t is a measurable subset of X. We first assume that $\mu(E_t) > 0$. We consider the family of all those $I \in \mathcal{T}$ maximal under the condition $Av_I(w) > c\lambda$, and denote it by $(I_j)_j$. Then $(I_j)_j$ is pairwise disjoint and $E_t = \cup I_j$. Additionally for every j and $I \in \mathcal{T}$ such that $I \supsetneq I_j$ we have that $\frac{1}{\mu(I)} \int\limits_I w d\mu = Av_I(w) \leq c\lambda$ because of the maximality of I_j. In view of Lemma 1 this gives $I_j \in S_w = S$, for every j.

For every I_j consider $I_j^* \in S$. Then by Lemma 2, $y_{I_j} \leq [k - (k-1)c^{-1}] y_{I_j^*}$. By the above discussion we now have $y_{I_j^*} \leq c\lambda$. Thus we obtain as a consequence that

$$y_{I_j} \leq [k - (k-1)\delta]c\lambda = (kc - k + 1)\lambda, \quad \text{for every} \quad j.$$

This gives

$$\begin{aligned} \int_{I_j} w d\mu \leq (kc - k + 1)\lambda\mu(I_j) &\Rightarrow \int_{E_t} w d\mu \leq (kc - k + 1)\lambda\mu(E_t) \\ &\Rightarrow \frac{1}{\mu(E_t)} \int_{E_t} w d\mu \leq (kc - k + 1)\lambda. \qquad (3.1) \end{aligned}$$

Since $\mathcal{M}_{\mathcal{T}}w \leq cw$ μ-a.e on X, and $E_t = \{\mathcal{M}_{\mathcal{T}}w > c\lambda\}$ we obviously have $E_t \subseteq \{w > \lambda\} \cup H = \{w > w^*(t)\} \cup H$, where H is suitable subset of X with $\mu(H) = 0$.

There exist now $E_t^* \subseteq (0,1]$ Lesbesgue measurable such that $|E_t^*| = \mu(E_t) =: t_1$, and such that $\int\limits_{E_t^*} w^*(y)dy = \int\limits_{E_t} wd\mu$. By the equimeasurability of w and w^*, we can choose the set E_t^* such that $E_t^* \subseteq \{w^* > w^*(t)\} \subseteq (0,t)$. As an immediate consequence $t_1 \leq t$.

Since now $\mathcal{T}$ differentiates $L^1(X,\mu)$ we have that μ-almost every element of the set $\{w > c\lambda\} \subseteq X$ belongs to E_t. Since $\mu(E_t) > 0$ we also have that $\mu(\{w > c\lambda\}) > 0$. Let now t_2 be such that

$$w^*(t) > \lambda c \;\text{ for every }\; t \in (0,t_2) \;\text{ and }\; w^*(t) \leq c\lambda, \;\text{ for every }\; t \in (t_2, 1).$$

By the definition of E_t^* we have that $E_t^* = (0,t_2) \cup A_t$, where A_t is a Lesbesgue measurable subset of (t_2,t) and $|A_t| = t_1 - t_2$ (Of course $t_2 = |(0,t_2)| = |\{w^* > \lambda c\}| = \mu(\{w > \lambda c\}) \leq \mu(\{\mathcal{M}_{\mathcal{T}}w > \lambda c\}) = \mu(E_t) =: t_1$).

We will now prove the following inequality

$$\frac{1}{\mu(E_t)} \int_{E_t} wd\mu \geq \frac{1}{t} \int_0^t w^*(y)dy, \tag{3.2}$$

(3.2) is equivalent to

$$\begin{aligned} \frac{1}{t_1} \int_{E_t^*} w^*(y)dy \geq \frac{1}{t} \int_0^t w^*(y)dy \Leftrightarrow t \int_0^{t_2} w^*(y)dy + t \int_{A_t} w^*(y)dy \\ \geq t_1 \int_0^{t_2} w^*(y)dy + t_1 \int_{t_2}^t w^*(y)dy \\ \Leftrightarrow (t - t_1) \int_0^{t_2} w^*(y)dy + t \int_{A_t} w^*(y)dy \\ \geq t_1 \int_{t_2}^t w^*(y)dy, \end{aligned} \tag{3.3}$$

We define $\Gamma_t = (t_2,t) \setminus A_t$. Then (3.3) becomes

$$\begin{aligned} (t - t_1) \int_0^{t_2} w^*(y)dy + (t - t_1) \int_{A_t} w^*(y)dy \geq t_1 \int_{\Gamma_t} w^*(y)dy \\ \Leftrightarrow (t - t_1) \int_{E_t^*} w^*(y)dy \geq t_1 \int_{\Gamma_t} w^*(y)dy. \end{aligned} \tag{3.4}$$

Additionally

$$\int_{E_t^*} w^*(y)dy = \int_{E_t} wd\mu > \mu(E_t) \cdot c\lambda = c\lambda \cdot t_1,$$

since E_t is the pairwise disjoint union of $(I_j)_j$. Thus if we prove that

$$\int_{\Gamma_t} w^*(y)dy \leq c\lambda(t-t_1), \tag{3.5}$$

we complete the proof of (3.2). But (3.5) is obvious since $w^*(y) \leq c\lambda$ on (t_2, t), $\Gamma_t \subseteq (t_2, t)$ and

$$|\Gamma_t| = |(t_2, t) \setminus A_t| = (t-t_2) - |A_t| = t - t_1.$$

We thus have proved that for every w $\mathcal{T}$-step function and t such that $\mu(\{\mathcal{M}_{\mathcal{T}} w > c \cdot w^*(t)\}) > 0$, the following inequality is true

$$\frac{1}{t}\int_0^t w^*(y)dy \leq (kc-k+1)w^*(t). \tag{3.6}$$

If t is such that $\mu(\{\mathcal{M}_{\mathcal{T}} w > cw^*(t)\}) = 0$ then obviously $\mathcal{M}_{\mathcal{T}} w(x) \leq cw^*(t)$, for μ-almost every $x \in X$, so since $\mathcal{T}$ differentiates $L^1(X,\mu)$: $w(y) \leq cw^*(t)$ for almost all $y \in X$. This obviously gives (3.6), since $c \leq kc-k+1$.

Additionally if w is in general an A_1-weight with respect to $\mathcal{T}$, then an approximation argument by $\mathcal{T}$-simple A_1-weights gives the result for w. More precisely one can easily see, that if w is a A_1 weight with respect to $\mathcal{T}$, with A_1-constant $\left[w\right]_1^{\mathcal{T}} = c$ then there exists an increasing sequence of $\mathcal{T}$-simple functions, $(w_n)_n$, such that $w_n \leq w$ and $\left[w\right]_1^{\mathcal{T}} = c_n \leq c$ with the additional properties $w_n \to w$, μ a.e. and $c_n \to c$ as $n \to +\infty$. In order to finish the proof of Theorem 3 we just need to prove the sharpness of the result. We proceed to this as follows

Fix $k \geq 2$. We suppose that we are given a tree $\mathcal{T}$ of homogeneity k, and consider $\mathcal{T}_{(2)}$. Then

$$\mathcal{T}_{(2)} = \{P_1, \ldots, P_k, P_{k+1}, \ldots, P_{2k}, \ldots, P_{k^2-k+1}, \ldots, P_{k^2}\} \text{ where}$$

$$\mathcal{T}_{(1)} = \left\{ \bigcup_{i=1}^{k} P_i, \bigcup_{i=k+1}^{2k} P_i, \ldots, \bigcup_{i=k^2-k+1}^{k^2} P_i \right\} = \{I_1, I_2, \ldots, I_k\}.$$

We have that $\mu(P_i) = \frac{1}{k^2}$, $\forall\, i$.

Suppose $\delta > 0$ is such that $\delta < \frac{1}{k^2}$, and consider for any such δ a set A_δ of measure $\mu(A_\delta) = \delta$ such that $A_\delta \subseteq P_1$ ((X,μ) is non atomic). Let $c \geq 1$ and $\alpha, \epsilon > 0$ be such that $\epsilon < \alpha$ and $kc - k + 1 = \frac{\alpha}{\epsilon}$. Let $\varphi = \varphi_\delta$ be the function defined as follows:

$$\begin{aligned}
&\varphi/A_\delta := \alpha \\
&\varphi/I_1 \setminus A_\delta := \epsilon \\
&\varphi/P_{k+1} := \alpha, \qquad \varphi/(I_2 \setminus P_{k+1}) := \epsilon \\
&\varphi/P_{2k+1} := \alpha, \qquad \varphi/(I_3 \setminus P_{2k+1}) := \epsilon \\
&\qquad\qquad\qquad\qquad \cdots \\
&\varphi/P_{k^2-k+1} := \alpha, \quad \varphi/(I_k \setminus P_{k^2-k+1}) := \epsilon
\end{aligned}$$

It is easy to see that $\varphi = \varphi_\delta$ is a A_1 weight with A_1 constant

$$c_\delta = \big[\varphi\big]_1^{\mathcal{T}} = \frac{Av_{I_2}(\varphi)}{\epsilon} = \frac{k}{\epsilon}\int_{I_2} \varphi d\mu = \frac{k}{\epsilon}\left[a\frac{1}{k^2} + \left(\frac{1}{k} - \frac{1}{k^2}\right)\epsilon\right].$$

Then $c_\delta = c$, is independent of δ. Additionaly $\varphi_\delta^*(1/k) = \epsilon$, so $\varphi_\delta^*(1/k)(kc-k+1) = \alpha$, while $k\int_0^{1/k} \varphi_\delta^*(y)dy$ tends to α, as $\delta \to 1/k^{2^-}$.

By this we end the proof of Theorem 3.

Theorem 1 of Section 1 is an immediate Corollary of Theorem 3. Additionally the following are consequences of Theorem 3.

Corollary 1. *Let w be an A_1 weight with respect to the tree $\mathcal{T}$ of homogeneity $(k \geq 2)$ on (X, μ) with $\big[w\big]_1^{\mathcal{T}} = c$. Then if one considers $((0,1], |\cdot|)$ equipped with the usual k-adic tree $\mathcal{T}_k$, where $|\cdot|$ is the Lesbesgue measure on $(0,1]$. Then $\big[w^*\big]_1^{\mathcal{T}_k} \leq kc - k + 1$ and this result is sharp.*

Proof. The proof is obvious. We just need to consider the function φ_δ constructed at the end of Theorem 3.

Corollary 2. *Let w be A_1-weight on $\mathbb{R}^n$ as described in Section 1. Then $w^* : (0, +\infty) \to \mathbb{R}^+$ has the following property:*

$$\frac{1}{t}\int_0^t w^*(y)dy \leq (kc - k + 1)w^*(t), \quad \textit{for every} \ \ t \in (0, +\infty)$$

and the last inequality is sharp.

Proof. We expand $\mathbb{R}^n$ as a union of an increasing sequence $(Q_j)_j$ of dyadic cubes, and use Theorem 3 in any of these.

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References

[1] B. Bojarski, C. Sbordone and I. Wik, The Muckenhoupt class $A_1(\mathbb{R})$, *Studia Math.* **101** (2) (1992), 155-163.

[2] R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241-250.

[3] F. W. Gehring, The L^p integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* **130** (1973), 265-277.

[4] J. Kinnunen, Sharp results on reverse Hölder inequalities, *Ann. Acad. Sci. Fenn. Ser A I Math. Diss* **95** (1994), 1-34.

[5] J. Kinunnen, A stability result on Muckenhoupt weights, *Publ. Math.* **42** (1998), 153-163.

[6] E.Y.Leonchik, On an estimate for the rearrangement of a function from th Muckenhoupt class A_1, *Ukrainian Math. J.* **62** (2011), no. 8, 1333-1338.

[7] A. Melas, A sharp L^p inequality for dyadic A_1 weights in $\mathbb{R}^n$, *Bull. London Math. Soc.* **37** (2005), 919-926.

[8] A. Melas, The Bellman functions of dyadic-like maximal operators and related inequalities, *Adv. in Math.* **192** (2005), 310-340.

[9] B. Muckenhoupt, Weighted norm inequalities for the Hardy-Littlewood maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207-226.

[10] A.Osekowski, Sharp inequalities for dyadic A_1 weights, *Arch Math. (Basel)* **101** (2013), no. 2, 181-190.

[11] V.I.Vasyunin, The exact constant in the inverse Hölder inequality for Muckenhoupt weights (Russian), *Algebra i Analiz* **15** (2003), no. 1, 73-117. Translation in *St. Petersburg Math. J.* **15** (2004), no.1, 49-79.

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