The eigenvalue problem for a bianisotropic cavity

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We consider the most general case of a linear medium where the constitutive relations are

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ε, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$: 3×3 matrices, having as entries complex functions of the position vector r and the angular frequency ω.

In the six vector notation, the problem is stated as follows:

$$\mathrm{i}\omega \begin{bmatrix} \varepsilon & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{bmatrix} \mathrm{e} = \begin{bmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{bmatrix} \mathrm{e}, \tag{3}$$

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Remark: The matrix in the left hand side depends on the eigenvalue ω.

The formal eigenvalue problem under consideration is written as

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$$\mathcal{Q} := \mathrm{i} \begin{bmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{bmatrix}$$

is the (formally self-adjoint) Maxwell operator.

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We further assume that the "wall" $\Gamma := \partial\Omega$ is perfect conducting,

Assumption 2

$\hat{n} \times E = \mathbf{0}$ on Γ.

The domain of $\mathcal{Q}$ is

$$D(\mathcal{Q}) := H_0(\mathrm{curl}\,; \Omega) \times H(\mathrm{curl}\,; \Omega).$$

and $\mathcal{X} := L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^2)$.

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$H^1(\Omega) \hookrightarrow L^2(\Omega)$ *with a compact injection.*

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$(\omega_n^0)_{n\in\mathbb{N}^*}$ is an increasing sequence of non-negative numbers, thus diverging at infinity, and $\omega_{-n}^0 = -\omega_n^0$.

Each ω_n^0, $n \neq 0$, is counted as many times as its multiplicity.

$\omega_0^0 := 0$ is always an eigenvalue but needs a special treatment, since the kernel $\ker \mathcal{Q}$ is infinite dimensional.

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Remark: Eigenvectors corresponding to different eigenvalues are orthogonal, so $(e_n^0)_{n \in \mathbb{Z}^*}$ can be chosen as an orthonormal sequence.

Let now $\mathcal{H} := \overline{[..., e_{-n}, ..., e_{-2}, e_{-1}, e_1, e_2, ..., e_n, ...]}$.
The restriction of $\mathcal{Q}$ on $\mathcal{H}$ is denoted by $\mathcal{Q}_{\mathcal{H}}$. $\mathcal{Q}_{\mathcal{H}}$ is both the restriction and the part of $\mathcal{Q}$ on $\mathcal{H}$. Actually, it is the spectrum of $\mathcal{Q}_{\mathcal{H}}$ that is discrete and one can strictly prove that.

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Proposition 3

$\mathcal{Q}_{\mathcal{H}}$ is a self-adjoint operator in $\mathcal{H}$ and has compact inverse $\mathcal{Q}_{\mathcal{H}}^{-1}$. The sequence of eigenvalues of $\mathcal{Q}_{\mathcal{H}}$ is $(\omega_n^0)_{n\in\mathbb{Z}^*}$ and the sequence of the corresponding eigenvectors is $(e_n^0)_{n\in\mathbb{Z}^*}$. The latter is an orthonormal basis for $\mathcal{H}$.

Then $(\star)$ is restricted in $\mathcal{H}$ and is written $\mathsf{e} = \omega \mathcal{Q}_{\mathcal{H}}^{-1} \mathsf{M}(\omega)\mathsf{e}$.

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Let $\mathcal{F}(\omega) := \omega \mathcal{Q}_{\mathcal{H}}^{-1} \mathsf{M}(\omega)$; we then conclude to the eigenvalue problem for a linear pencil

$$(I - \mathcal{F}(\omega))\mathsf{e} = 0 \tag{4}$$

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$\mathcal{F}(\omega)$: is a compact operator in $\mathcal{X}$.

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Definition 2

An ω is called eigenvalue of the pencil $I - F(\cdot)$ if the equation $F(\omega)x = x$ has non trivial solutions. A non trivial solution of $F(\omega)x = x$ $\omega \in S$, is called an eigenvector corresponding to ω and the linear span of the eigenvectors is called the eigenspace corresponding to ω.

We have the following important result

Proposition 4 (Analytic Fredholm Alternative)

Let F be analytic and $F(\omega) \in \mathcal{K}(X)$ for all $\omega \in D$. Then either

a) $I - F(\omega)$ is not injective for every $\omega \in D$,

or

b) $(I - F(\omega))^{-1} \in \mathcal{B}(X)$ for all $\omega \in D \backslash S$, where $S \subset \mathbb{C}$ is a countable set without any limit point.

Remark: In case (b), the operator-valued function $(I - F(\cdot))^{-1}$ is analytic in $D \backslash S$, meromorphic in D and the residues at the poles are finite rank operators.

We now focus in the special case $F(\omega) := AB(\omega)$, $\omega \in D$, where A is compact self-adjoint and $B(\omega) \in \mathcal{B}(X)$. We treat the equation in an abstract sense in an arbitrary seperable Hilbert space X.

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The spectral theorem ensures that A is represented as

$$Ax = \sum_n \lambda_n \left\langle x, e_n^0 \right\rangle e_n^0, \tag{5}$$

where (λ_n) is the sequence of (non-zero real) eigenvalues of A, in an absolutely descending order and counted as many times as their multiplicity, and (e_n^0) is the sequence of corresponding eigenvectors.

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The latter is an orthonormal basis for $\overline{R(A)}$. Here we assume that A has infinitely many eigenvalues and thus $\lambda_n \to 0$.

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Let $f_n = f_n(\omega) := \left(B(\omega) - \frac{1}{\lambda_n} I\right)^* e_n^0 = \left(B(\omega)^* - \frac{1}{\lambda_n} I\right) e_n^0$. The LHS of (6) is a multiplier operator

$$S = S(\omega) := \sum_n \lambda_n \left\langle \cdot, f_n \right\rangle e_n^0,$$

corresponding to sequences $(\lambda_n) \subset \mathbb{R}$, $(f_n) \subset X$, $(e_n) \subset X$, where (λ_n) is bounded, (f_n) is a sequence and (e_n) is an orthonormal basis.

Proposition 5

The following are equivalent:
a) S is injective.
b) (f_n) is a complete sequence, i.e., $\overline{[f_1, f_2, ..., f_n, ...]} = X$.
c) $\langle x, f_n \rangle = 0$ for every n, implies $x = 0$.

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Corollary 3

ω *is an eigenvalue of* $I - F(\cdot)$ *if and only if the sequence* $(f_n(\omega))$ *is not complete. The corresponding eigenspace is* $\ker S(\omega)$*. Moreover,* $x \in \ker S(\omega)$ *if and only if* $\langle x, f_n(\omega) \rangle = 0$ *for every* n *and, consequently,*

$$\ker S(\omega) = [f_1(\omega), f_2(\omega), ..., f_n(\omega), ...]^{\perp} .$$

Let us now consider the inverse problem, that is to reconstruct the operator $B(\cdot)$ from the knowledge of the eigenelements of the problem $F(\omega)x = x$. Actually, let us assume that ω is an eigenvalue with corresponding eigenspace $\ker S(\omega)$. Then

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The straightforward relation

$$\left\langle B(\omega)e_n^0, e_m^0 \right\rangle = \left\langle e_n^0, f_m(\omega) \right\rangle + \frac{\delta_{nm}}{\lambda_n}, \tag{7}$$

(δ_{nm} stands for the Kronecker delta) allows the recovery of the operator $B(\omega)$.

We now return to problem

$$(I - \mathcal{F}(\omega))\mathsf{e} = 0 \tag{8}$$

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M *is an analytic function* $D \ni \omega \mapsto \mathsf{M}(\omega) \in \mathcal{B}(\mathcal{X})$*, where* D *is a domain in the complex plane, such that* $0 \in D$*.*

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Consequently, $\mathcal{F}$ defines an analytic function $D \ni \omega \mapsto \mathcal{F}(\omega) \in \mathcal{K}(\mathcal{X})$.

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Consequently, $\mathcal{F}$ defines an analytic function $D \ni \omega \mapsto \mathcal{F}(\omega) \in \mathcal{K}(\mathcal{X})$.

Moreover, for $\omega_0 = 0$, $\mathcal{F}(\omega_0) = 0$ and thus $I - \mathcal{F}(\omega_0)$ is invertible.

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The eigenvectors corresponding to an eigenfrequency ω_n are called the corresponding modes.

Let now $\mathcal{A} := \mathcal{Q}_{\mathcal{H}}^{-1}$, $\mathsf{B}(\omega) := \omega \mathsf{M}(\omega)$.
The eigenvalues of $\mathcal{A}$ are calculated as follows

$$\lambda_n := \frac{1}{\omega_n^0} \ , \ n = \pm 1, \pm 2, ...,$$

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Consequently, one has the expansion

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Now $\mathcal{F}(\omega) = \mathcal{A}\mathsf{B}(\omega)$ and it is reformulated with the aim of the multiplier operator

$$\mathcal{S}\mathsf{x} = \mathcal{S}(\omega)\mathsf{x} := \sum_{n=-\infty}^{\infty} \lambda_n \left\langle \mathsf{x}, \mathsf{f}_n \right\rangle \mathsf{e}_n^0,$$

where

$$\mathsf{f}_n = \mathsf{f}_n(\omega) := \left(\mathsf{B}(\omega) - \frac{1}{\lambda_n} I\right)^* \mathsf{e}_n^0 = \left(\bar{\omega}\mathsf{M}(\omega)^* - \omega_n^0 I\right) \mathsf{e}_n^0 \ , \ n = \pm 1, \pm 2, ...$$

Proposition 7

$\omega \neq 0$ is an eigenfrequency of the cavity if and only if $\mathcal{S}(\omega)$ is not an injective operator, if and only if $(\mathrm{f}_n(\omega)_{n\in\mathbb{Z}^*}$ is not complete. The corresponding subspace of modes is finite dimensional and is given by

$$\ker \mathcal{S}(\omega) = [, ..., \mathrm{f}_{-n}(\omega), ..., \mathrm{f}_{-2}(\omega), \mathrm{f}_{-1}(\omega), \mathrm{f}_1(\omega), \mathrm{f}_2(\omega), ..., \mathrm{f}_n(\omega), ...]^{\perp} .$$

Moreover, the following equality applies

$$\left\langle \mathrm{M}(\omega)\mathrm{e}_n^0, \mathrm{e}_m^0 \right\rangle = \omega_n^0 \left\langle \mathrm{e}_n^0, f_m(\omega) \right\rangle + \delta_{nm},$$

from which the material matrix can be recovered.

Thank you!